EERIN
RARY

JUN 21 1950

THE AERONAUTICAL QUARTERLY

Volume II MAY 1950 Part I

CONTENTS

Two-Dimensional Theory of Stiffened Plates — *Chang O'Chou*

The Whirling of a Spinning Top — *J. Morris*

The Possibility of the Determination of Rate of Climb from Acceleration Measurements in Level Flight — *E. C. Pike*

A Note on Subsonic Aerofoil Theory — *John W. Miles*

Theory of an Oscillating Supersonic Aerofoil — *Geoffrey L. Sewell*

The Linearised Theory of Conical Fields in Supersonic Flow, with Applications to Plane Aerofoils — *S. Goldstein* and *G. N. Ward*

LONDON
ROYAL AERONAUTICAL SOCIETY
4 HAMILTON PLACE W1

10/-

THE AERONAUTICAL QUARTERLY

Editorial Board

W. G. A. PERRING, F.R.Ae.S. (*Chairman*)

Dr. H. ROXBEE COX, D.I.C., F.R.Ae.S., F.I.Ae.S.

Professor S. GOLDSTEIN, F.R.S., F.R.Ae.S.

Sir GEOFFREY TAYLOR, F.R.S.

G. H. DOWTY, F.R.Ae.S. (Chairman of the JOURNAL Committee of the Royal Aeronautical Society)

J. SMITH, C.B.E., F.R.Ae.S., A.M.I.Mech.E. (Chairman of the Technical Board and Executive Committee of the Society of British Aircraft Constructors)

Joint Editors

Captain J. Laurence Pritchard, Hon.F.R.Ae.S., Hon.F.I.Ae.S.
Joan Bradbrooke, A.R.Ae.S. Dr. D. M. A. Leggett, A.F.R.Ae.S.

Editorial Referees

Aerodynamics

(a) *General*

Prof. L. BAIRSTOW
Dr. G. P. DOUGLAS
Prof. A. A. HALL
Sir B. MELVILL JONES
Mr. D. KEITH-LUCAS
Dr. N. A. V. PIERCY
Mr. W. G. A. PERRING
Mr. E. F. RELF
Mr. E. J. RICHARDS

(b) *Fluid Motion*

Mr. A. FAGE
Prof. S. GOLDSTEIN
Prof. L. HOWARTH
Mr. W. P. JONES
Dr. J. W. MACCOLL
Dr. J. H. PRESTON
Mr. L. F. G. SIMMONS
Mr. H. B. SQUIRE
Sir GEOFFREY TAYLOR
Prof. G. TEMPLE
Mr. A. D. YOUNG

(c) *Stability and Control*

Mr. L. W. BRYANT
Mr. S. B. GATES
Mr. G. T. R. HILL
Mr. H. B. IRVING
Mr. G. H. LEE
Mr. M. B. MORGAN

Helicopters and Propellers

Dr. J. A. J. BENNETT
Mr. L. G. FAIRHURST
Mr. R. HAFNER
Capt. R. N. LIPTROT

Instrumental and Electrical Equipment

Dr. G. E. BAIRSTO
Mr. G. W. H. GARDNER
Mr. C. S. HUDSON
Mr. A. N. IRENS
Mr. C. N. JAQUES
Mr. F. W. MEREDITH

Aircraft Design

Mr. R. BOORMAN
Mr. R. M. CLARKSON
Mr. A. N. CLIFTON
Mr. W. S. FARREN
Mr. H. H. GARDNER
Mr. H. KNOWLER
Mr. W. E. W. PETTER
Mr. F. H. POLLICUTT
Mr. A. E. RUSSELL
Mr. R. S. STAFFORD

Radio and Radar

Mr. J. F. ATHERTON
Mr. B. G. GATES
Mr. R. L. SMITH ROSE
Dr. A. M. UTTLEY
Sir ROBERT WATSON WATT

Aviation Medicine

Dr. A. BUCHANAN BARBOUR

Meteorology

Prof. D. BRUNT
Mr. J. K. HARDY
Prof. P. A. SHEPPARD

Aircraft Propulsion

Mr. H. CONSTANT
Sir A. H. ROY FEDDEN
Dr. S. HOOKER
Mr. P. LLOYD
Dr. A. W. MORLEY
Dr. E. S. MOULT
Mr. N. S. MUIR
Mr. F. NIXON
Mr. F. M. OWNER
Dr. D. PYE
Mr. H. SAMMONS
Prof. O. A. SAUNDERS

Air Transport

Mr. P. G. MASEFIELD
Mr. N. E. ROWE
Mr. W. TYE

Aircraft Accessories

Mr. W. C. CLOTHIER
Mr. H. G. CONWAY
Mr. G. H. DOWTY
Mr. R. GRAHAM

Vibration and Flutter

Prof. A. R. COLLAR
Prof. W. J. DUNCAN
Mr. R. A. FRAZER
Mr. J. HANSON
Mr. A. ROBINSON

Performance

Mr. H. DAVIES
Mr. D. R. NEWMAN
Mr. A. H. YATES

Structures

Mr. A. C. BROWN
Mr. H. L. COX
Mr. D. J. FARRAR
Dr. H. ROXBEE COX
Mr. W. S. HEMP
Mr. H. B. HOWARD
Mr. E. D. KEEN
Mr. F. A. KERRY
Dr. D. M. A. LEGGETT
Prof. R. L. LICKLEY
Prof. A. J. SUTTON PIPPARD
Prof. A. G. PUGSLEY
Dr. S. C. REDSHAW
Mr. R. H. SANDIFER
Sir RICHARD SOUTHWELL

Fuels and Oils

Mr. E. L. BASS
Mr. J. W. DRINKWATER
Prof. F. H. GARNER
Prof. W. E. GARNER
Dr. K. E. W. RIDLER
Dr. C. G. WILLIAMS

Materials

Prof. L. AITCHISON
Dr. N. P. ALLEN
Prof. N. F. MOTT
Mr. L. ROTHERHAM
Dr. H. SUTTON
Dr. C. SYKES

Flight Testing

Mr. P. A. HUFTON
Mr. E. T. JONES
Mr. S. SCOTT HALL
Mr. H. L. STEVENS

TWO DIMENSIONAL THEORY OF STIFFENED PLATES

by

CHANG O'CHOU, Ph.D.
(The University, Bristol)

SUMMARY

The two-dimensional stiffened plate problem is completely governed by the fundamental differential equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{1}{k_1^2}\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{1}{k_2^2}\frac{\partial^2}{\partial y^2}\right)\phi=0 \quad . \quad . \quad . \quad (22)$$

where k_1 and k_2 are two non-dimensional structural constants depending on the relative proportions of the reinforcing members.

The complete solution is

$$\phi=f_1(x+ik_1y)+f_2(x-ik_1y)+g_1(x+ik_2y)+g_2(x-ik_2y) \quad . \quad (25)$$

This solution renders all solutions for the isotropic plate available for the stiffened plate also. The choice between the several types of solution available is governed by the same considerations as for the isotropic or unstiffened plate.

1. INTRODUCTION

By the term "stiffened plate" is meant a plate of thickness t, reinforced by longitudinal members (stringers), each of section A_s, at spacing b, and transverse members (ribs), each of section A_r at spacing a, as shown in Fig. 1. Both longitudinal and transverse reinforcing members are continuously attached to the sheet, and are treated as of negligible stiffness in bending. Thus, along the line of attachment, the displacement is the same for the sheet and the reinforcing member. A rigorous analysis of such stiffened plates results in partial finite difference equations, the solution of which is not familiar at the present time. However, the general stress

Paper received January 1949.
[The Aeronautical Quarterly, Vol. II, May 1950]

distribution can be obtained by considering both sets of reinforcing members as uniformly distributed; in other words, as two hypothetical sheets with thickness $t_s = A_s/b$ for stringers and thickness $t_r = A_r/a$ for ribs. The hypothetical sheet differs from the actual sheet in that it can only transmit normal stress in the direction of the reinforcing members which it represents and it is discontinuous in the direction perpendicular to the reinforcing members.

Throughout this paper the x-axis is in the direction of the stringers and the y-axis in the direction of the ribs, as in Fig. 1.

Notation

A_r, A_s	area of section of ribs and stringers respectively
a, b	spacing of ribs and stringers respectively
x, y	rectangular co-ordinates (*see* Fig. 1)
u, v	displacements in x and y directions respectively
x_0, y_0	*see* Fig. 2
E	Young's modulus
σ	Poisson's ratio
f_x, f_y, q	normal components of stress, and shear stress in plate
d, l	dimensions defining the boundaries of a rectangular plate (*see* Section 3)
ξ	$\pi x/l$
η	$\pi y/d$
t	thickness of plate
t_r	A_r/a
t_s	A_s/b
α_x, α_y, α_{xy}	defined in equations (19), (20) and (21)
P, Q, R	defined by equations (14), (15) and (16)
ϕ	*see* equation (13)
χ	stress function
k_1, k_2	defined in equations (23) and (24)
k_1', k_2'	defined in equations (29) and (30)
K_1, K_2, K_3, K_4	constants of integration

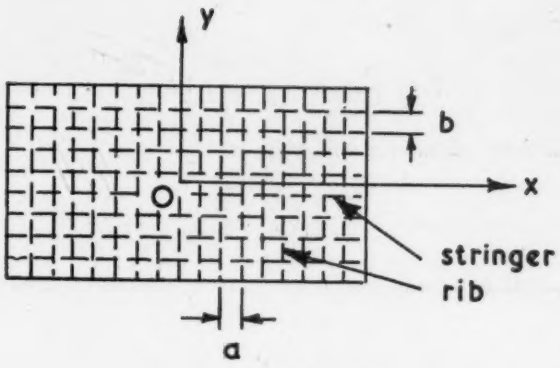

Fig. 1.

2. GENERAL ANALYSIS

Let u and v denote the displacements in the x and y directions respectively. Then

$$f_x \text{ in plate} = \frac{E}{1-\sigma^2}\left(\frac{\partial u}{\partial x} + \sigma\frac{\partial v}{\partial y}\right) \quad . \quad . \quad . \quad (1)$$

$$f_y \text{ in plate} = \frac{E}{1-\sigma^2}\left(\sigma\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \quad . \quad . \quad . \quad (2)$$

$$q = \frac{E}{2(1+\sigma)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad . \quad . \quad . \quad (3)$$

$$f_x \text{ in stringer sheet} = E\frac{\partial u}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$f_y \text{ in rib sheet} = E\frac{\partial v}{\partial y}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

f_x, f_y, q are respectively the normal components of stress and the shear stress in the plate.

Thus the total loading per unit width of stiffened plate in direction x is

$$f_x t = \frac{Et}{1-\sigma^2}\left(\frac{\partial u}{\partial x} + \sigma\frac{\partial v}{\partial y}\right) + Et_s\frac{\partial u}{\partial x} \quad . \quad . \quad . \quad . \quad (6)$$

and in direction y is

$$f_y\, t = \frac{Et}{1-\sigma^2}\left(\sigma\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+Et_r\frac{\partial v}{\partial y} \quad . \quad . \quad . \quad . \quad (7)$$

while the total shear loading per unit width is

$$qt = \frac{Et}{2(1+\sigma)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \quad . \quad . \quad . \quad . \quad . \quad (8)$$

By adopting these definitions of f_x and f_y, the conditions for equilibrium become

$$\left.\begin{aligned}\frac{\partial f_x}{\partial x}+\frac{\partial q}{\partial y}&=0\\ \text{and}\qquad \frac{\partial f_y}{\partial y}+\frac{\partial q}{\partial x}&=0.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

As usual, a stress function χ may be assumed such that

$$f_x = \frac{\partial^2\chi}{\partial y^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$f_y = \frac{\partial^2\chi}{\partial x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and

$$q = -\frac{\partial^2\chi}{\partial x\partial y}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Substituting (10), (11) and (12) into (6), (7) and (8), eliminating any two of χ, u and v, and using ϕ to represent the third, it follows that

$$\left(P\frac{\partial^4}{\partial x^4}+Q\frac{\partial^4}{\partial x^2\partial y^2}+R\frac{\partial^4}{\partial y^4}\right)\phi=0, \quad . \quad . \quad . \quad . \quad (13)$$

where

$$P = 1+\frac{A_s}{bt}(1-\sigma^2) \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$Q = \frac{2}{1-\sigma}(PR-\sigma) \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$R = 1+\frac{A_r}{at}(1-\sigma^2). \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Equation (13) is the fundamental differential equation for the stiffened plate.

For problems with given boundary loadings, it is more convenient to treat ϕ as representing χ. In this case u and v are given by

$$E\frac{\partial u}{\partial x}=\alpha_x\frac{\partial^2\chi}{\partial y^2}-\alpha_{xy}\frac{\partial^2\chi}{\partial x^2} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and

$$E\frac{\partial v}{\partial y}=\alpha_y\frac{\partial^2\chi}{\partial x^2}-\alpha_{xy}\frac{\partial^2\chi}{\partial y^2}, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where

$$\alpha_x=\frac{(1-\sigma^2)R}{(PR-\sigma^2)} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$\alpha_y=\frac{(1-\sigma^2)P}{(PR-\sigma^2)} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and

$$\alpha_{xy}=\frac{\sigma(1-\sigma^2)}{(PR-\sigma^2)}. \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The similarity between (17), (18) and the familiar stress-strain relations for isotropic plates is obvious. The three non-dimensional constants α_x, α_y and α_{xy} may be called the x-ratio, y-ratio and xy-ratio of the stiffened plate.

3. GENERAL SOLUTION OF THE FUNDAMENTAL EQUATION (13)

It can be proved that the discriminant Q^2-4PR of (13) is always positive. Thus (13) may be rewritten as

$$\left(\frac{\partial^2}{\partial x^2}+\frac{1}{k_1^{\,2}}\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{1}{k_2^{\,2}}\frac{\partial^2}{\partial y^2}\right)\phi=0 \quad . \quad . \quad . \quad (22)$$

with

$$1/k_1^{\,2}=Q/2+\sqrt{(Q^2/4-PR)} \quad . \quad . \quad . \quad . \quad (23)$$

and

$$1/k_2^{\,2}=Q/2-\sqrt{(Q^2/4-PR)}. \quad . \quad . \quad . \quad . \quad (24)$$

The general solution of (22) is

$$\phi=f_1(x+ik_1y)+f_2(x-ik_1y)+g_1(x+ik_2y)+g_2(x-ik_2y) \quad . \quad (25)$$

For stiffened plates with rectangular boundaries equation (25) may be expanded into double series with suitable periods, *i.e.*

$$\phi=\sum_{k_1',\,k_2'}\sum_{p=0}^{\infty}A_p\begin{Bmatrix}\sin p\xi\\ \cos p\xi\end{Bmatrix}\begin{Bmatrix}\cosh k_1'p\eta\\ \sinh k_1'p\eta\end{Bmatrix}+\sum_{k_1',\,k_2'}\sum_{p=0}^{\infty}A_p'\begin{Bmatrix}\sinh p\xi\\ \cosh p\xi\end{Bmatrix}\begin{Bmatrix}\sin k_1'p\eta\\ \cos k_1'p\eta\end{Bmatrix}$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

where
$$\xi = \pi x/l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$
$$\eta = \pi y/d, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$
$$k_1' = k_1 d/l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$
$$k_2' = k_2 d/l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and the four boundaries are $x = \pm l$ and $y = \pm d$. It is understood that A_p assumes different values for different combinations.

4. A SIMPLE EXAMPLE

Consider for example a rectangular stiffened plate under pure bending in its own plane. Let

$$\chi = y^3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

which is a solution of (13). Then

$$f_x = 6y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$
$$f_y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$
$$q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$
$$Eu = 6\alpha_x xy + K_1 + K_2 y \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$
$$Ev = -3\alpha_{xy} y^2 + K_3 + K_4 x \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Terms containing K_1, K_2, K_3, K_4 represent rigid body displacements and can therefore be ignored.

Both u and v are single-valued everywhere; hence (31) is a solution. From (1), (2), (4), (5) and (12) it follows that

$$f_x \text{ in plate} = \frac{6\alpha_x}{1-\sigma^2}\left(1 - \frac{\sigma^2}{R}\right) y \quad . \quad . \quad . \quad . \quad (37)$$
$$f_y \text{ in plate} = \frac{6\alpha_{xy}}{1-\sigma^2}(R-1)\, y \quad . \quad . \quad . \quad . \quad (38)$$
$$f_x \text{ in stringer sheet} = 6\alpha_x y \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$
$$f_y \text{ in rib sheet} = -6\alpha_{xy} y \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

and
$$q = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Along the boundaries $y = \pm d$, both f_y and q vanish. There is therefore no resultant force acting anywhere along the horizontal boundaries.

Along the boundaries $x = \pm l$, shear vanishes, while normal stress f_x is proportional to the distance from the x-axis, *i.e.* the distribution is linear. The resultant force acting on the vertical boundary is zero while the resultant moment is

$$M = \int_{-d}^{d} 6y^2 t dy = 4td^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

The stiffened plate is thus in a state of pure bending under this couple. The sign convention is such that a clockwise moment is positive at the right hand end.

5. ANOTHER EXAMPLE

Another example is a semi-infinite plate under a concentrated force acting perpendicular to the boundary. Let

$$\chi = (x + ik_1 y)\log(x + ik_1 y) + (x - ik_1 y)\log(x - ik_1 y) - \\ - (x + ik_2 y)\log(x + ik_2 y) - (x - ik_2 y)\log(x - ik_2 y) \quad . \quad (43)$$

which is a solution of (13). Then

$$f_x = -\frac{2k_1^2 x}{x^2 + k_1^2 y^2} + \frac{2k_2^2 x}{x^2 + k_2^2 y^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

$$f_y = \frac{2x}{x^2 + k_1^2 y^2} - \frac{2x}{x^2 + k_2^2 y^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

$$q = -\frac{2k_1^2 y}{x^2 + k_1^2 y^2} + \frac{2k_2^2 y}{x^2 + k_2^2 y^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

$$Eu = -(k_1^2 \alpha_x + \alpha_{xy})\log(x^2 + k_1^2 y^2) + (k_2^2 \alpha_x + \alpha_{xy})\log(x^2 + k_2^2 y^2) \quad . \quad . \quad (47)$$

$$Ev = [2(\alpha_y + k_1^2 \alpha_{xy})/k_1]\tan^{-1}(k_1 y/x) - [2(\alpha_y + k_2^2 \alpha_{xy})/k_2]\tan^{-1}(k_2 y/x) \quad . \quad (48)$$

In general, there is a line of discontinuity of v along the y-axis. Consequently, (43) is not a solution for plates containing the y-axis. However, both u and v are single-valued either in the right half plane or in the left half plane of the y-axis; therefore (43) is a solution for the semi-infinite plane or any portion of it subjected to the edge stresses dictated by the stress formulae.

Take the right half plane as an example; all boundary stresses vanish except at the origin which is a singular point. To evaluate the total force acting at the

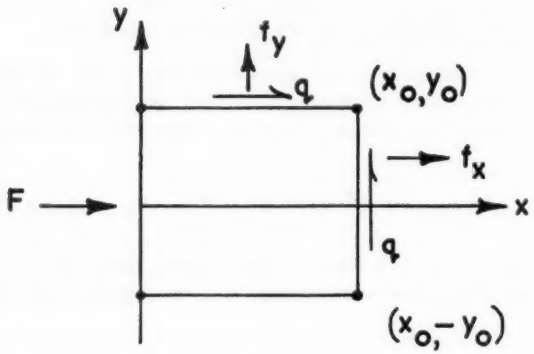

Fig. 2.

origin, the condition of equilibrium of the portion of the plate including the origin is applied, as shown in Fig. 2.

Because of symmetry, F is given by

$$F = -2t \int_0^{x_0} q_{(y=y_0)}\, dx - 2t \int_0^{y_0} f_{x\,(x=x_0)}\, dy$$
$$= 2\pi t (k_1 - k_2) \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Therefore, this example corresponds to the classic radial pressure case for the isotropic plate and shows how the stress distribution is distorted by the presence of reinforcing members. For plates stiffened in the x direction only, this problem is usually classified under the heading "load diffusion."

ACKNOWLEDGMENT

The author wishes to express his appreciation to Mr. H. L. Cox for valuable suggestions and criticisms.

THE WHIRLING OF A SPINNING TOP

by

J. MORRIS

(Consultant, Structures and Mechanical Engineering Departments, Royal Aircraft Establishment)

SUMMARY

This paper has been compiled because of the applicability of the treatment of the stability of the motion of the common spinning top to the problem of the whirling of rotating shafts, carrying loads of appreciable moments of inertia. This problem is assuming renewed interest and importance, especially in the drives of contra-propeller systems and the more recent high speed prime movers.

1. INTRODUCTION

This investigation deals with an old problem partly from a new angle. The reason for the resuscitation of the treatment of the stability of a common spinning top is because the analysis for the elucidation of whirling phenomena in rotating shaft systems is analogous to the classical analysis of the disturbed motion of a top. Whirling has become once again a problem of current and urgent importance primarily in pursuance of the recent development of light weight prime movers whose rotors revolve at very high speeds and secondly in connection with the mechanism for operating contra-propeller systems. Apart from these considerations the present treatment would appear to merit attention on its own account, as the question of the whirling of a top in consequence of unbalance does not appear to have been probed previously. It will be shown that the ordinary symmetrical top may have only one whirling speed and this will be above the minimum speed for stability in the vertical position. The unsymmetrical top however, in certain circumstances, may have two whirling speeds between which the unforced disturbed motion is exponentially unstable; and in general the conditions for its stability are somewhat more complicated than for the symmetrical top.

Lanchester in Appendix VII of his " Aerodonetics " remarks

" It is usual to treat the gyroscope and problems involving gyroscopic actions exclusively by mathematical methods in which the physical reality of the problem is partially obscured. One result of this is that, in spite of the unimpeachable nature

Paper received November 1949.

[The Aeronautical Quarterly, Vol. II, May 1950]

of the said methods, mistakes are not infrequently made, even the direction of the gyroscopic torque is sometimes misstated."

Discussing the gyroscope precessing in a horizontal plane, Lanchester gives the correct result for the gyroscopic torque, namely $\tau = I\omega\Omega$ where I, ω are respectively the moment of inertia and the angular velocity about the principal axis, and Ω is the angular velocity of precession. But when he proceeds to deal with the case when the axis of the gyroscope precesses at an angle α to the vertical, Lanchester derives the incorrect result namely, $\tau = I\omega\Omega \sin\alpha$, thus commiting a curious error in his physical reasoning concerning a basic proposition in gyro-statics.

2. KINEMATICS OF THE PROBLEM

The problem considered is that of a top spinning on a perfectly rough horizontal plane.

Referring to Fig. 1, let O be the point of contact with the ground of the peg of the top, of which OC is the principal axis of rotation, and let OA, OB, be the two other principal axes at O. It is assumed that the top is maintained in constant angular rotation about the axis OC, which is regarded as executing vibrations of small amplitude about OZ, a fixed vertical axis. OA, OB, OC, are a system of rectangular axes which are principal axes fixed in the top; ZN is an arc drawn perpendicular to the quadrant BC produced and meeting it at N. The arc ZN is taken to be Φ and the arc NC to be Θ, both these arcs being supposed small. Let the constant angular velocity imposed on the top be Ω; thus designating the three angular velocities of the axes OA, OB, OC, respectively as ω_1, ω_2, ω_3, we have, since the arcs Φ, Θ, are by supposition, small and are rotating with constant angular velocity Ω about OZ at Z,

$$\omega_1 = -(\dot{\Theta} + \Omega\Phi) \text{ along } BC, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\omega_2 = (\Phi - \Omega\Theta) \text{ along } CA, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\omega_3 = \Omega \text{ in the direction } AB. \quad . \quad . \quad . \quad . \quad (3)$$

G is the centre of gravity of the top, at the distance l from 0.

3. DYNAMICS OF THE PROBLEM

Let A, B, C, be the principal moments of inertia of the top at the point O. The inertia moments about the corresponding axes at O, fixed in the top, will be

$$[A\dot{\omega}_1 + (C - B)\omega_2\omega_3]$$

or

$$-A\ddot{\Theta} - (C - B)\Omega^2\Theta - (A + B - C)\Omega\dot{\Phi} \quad . \quad . \quad . \quad . \quad (4)$$

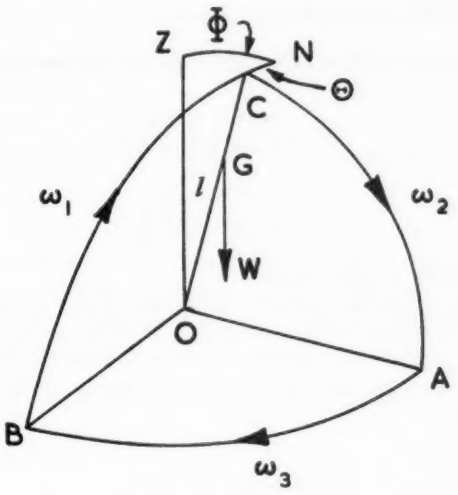

Fig. 1.

about the, axis OA in the direction C to B; and similarly

$$-[B\dot{\omega}_2-(C-A)\,\omega_1\,\omega_3]$$

or
$$-B\ddot{\Phi}-(C-A)\,\Omega^2\Phi+(A+B-C)\,\Omega\dot{\Theta} \quad . \quad . \quad . \quad . \quad (5)$$

about the axis OB in the direction C to A. Hence the equations of motion will be

$$-a\ddot{\Theta}-(c-b)\,\Omega^2\Theta-(a+b-c)\,\Omega\dot{\Phi}+Wl\Theta+mld\,\Omega^2\sin\varepsilon_d=0, \quad . \quad (6)$$

$$-b\ddot{\Phi}-(c-a)\,\Omega^2\Phi+(a+b-c)\,\Omega\dot{\Theta}+Wl\Phi+mld\,\Omega^2\cos\varepsilon_d=0, \quad . \quad (7)$$

where in engineers' units W is the weight of the top, $m=W/g$, $a=A/g$, etc., g being the acceleration due to gravity, and $mld\Omega^2$ the moment of dynamic unbalance (arising, for example, if OC is not precisely a principal axis), about an axis through O in the plane AOB. Let the axis of this unbalanced couple be inclined at an angle ε_d to OB in the direction A to B. The unbalance is assumed to be insufficient to have any appreciable effect on the principal moments of inertia of the top; it does however give rise to small products of inertia through the agency of which the unbalanced forces rotating with the impressed constant angular velocity become manifest as forced vibrations.

4. STEADY MOTION SOLUTION

Clearly a particular solution of equations (6) and (7) presents itself of the form

$$\Theta_0 = \frac{mld\,\Omega^2 \sin \varepsilon_d}{[(c-b)\Omega^2 - Wl]}, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\Phi_0 = \frac{mld\,\Omega^2 \cos \varepsilon_d}{[(c-a)\Omega^2 - Wl]}, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

which represents steady circular motion.

The speeds for which the radius or amplitude of this steady motion become, in the absence of damping, infinite, are called whirling speeds. Thus if a, b, are both greater than c, then Θ_0, Φ_0, can never become infinite for any real values of Ω. There are therefore in such cases no real whirling speeds. If, however, Ω becomes infinitely large then

$$\Theta_0 \text{ tends to } \frac{mld \sin \varepsilon_d}{(c-b)}, \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and

$$\Phi_0 \text{ tends to } \frac{mld \cos \varepsilon_d}{(c-a)}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If a, b, are both less than c, then there will be two real whirling speeds, namely

$$\Omega_1^2 = Wl/(c-b), \quad \Omega_2^2 = Wl/(c-a) \quad . \quad . \quad . \quad . \quad (12)$$

Clearly in this case if $a > b$ then $\Omega_2^2 > \Omega_1^2$. For the case $a > c > b$ there will be only one real whirling speed, namely that given by Ω_1^2 above.

5. STABILITY OF THE STEADY MOTION

Next the stability of the steady motion is investigated, which as has been seen is conditioned by the lack of balance of the top. Ignoring this unbalance, a solution is assumed for the free vibrations of the form

$$\Theta = \Theta_0 \sin(\omega t + \varepsilon), \quad \Phi = \Phi_0 \cos(\omega t + \varepsilon).$$

Inserting these values in the equations of motion, we obtain

$$[a\omega^2 - (c-b)\Omega^2 + Wl]\Theta_0 + (a+b-c)\Omega\omega\Phi_0 = 0, \quad . \quad . \quad . \quad (13)$$

$$(a+b-c)\Omega\omega\Theta_0 + [b\omega^2 - (c-a)\Omega^2 + Wl]\Phi_0 = 0. \quad . \quad . \quad . \quad (14)$$

On elimination of Θ_0 and Φ_0, we derive the frequency equation,

$$[a\omega^2+(b-c)\Omega^2+Wl][b\omega^2+(a-c)\Omega^2+Wl]-(a+b-c)^2\Omega^2\omega^2=0; \quad . \quad (15)$$

or

$$ab\omega^4-\{[ab+(b-c)(a-c)]\Omega^2-(a+b)Wl\}\omega^2+[(b-c)\Omega^2+Wl][(a-c)\Omega^2+Wl]=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

For stability of the disturbed steady motion the two roots in ω^2 of (16) must both be real and positive. Hence if there are two real whirling speeds, then apart from other considerations, the free vibrations are exponentially unstable for speeds between these whirling speeds; or if there is only one real whirling speed then stability is impossible for speeds above it.

Furthermore $$\{ab+(b-c)(a-c)\}\Omega^2-(a+b)Wl \quad . \quad . \quad . \quad . \quad (17)$$

must be positive; and

$$[\{ab+(b-c)(a-c)\}\Omega^2-(a+b)Wl]^2- -4ab[(b-c)(a-c)\Omega^4+(a+b-2c)Wl\Omega^2+W^2l^2] \quad . \quad . \quad (18)$$

must be positive; this latter condition may be expressed as

$$[c(a+b-c)\Omega^2-\{(4ab/c)-(a+b)\}Wl]^2>(4ab/c^2)(2a-c)(2b-c)W^2l^2. \quad (19)$$

6. THE SYMMETRICAL TOP

For the ordinary symmetrical top $b=a$; in which case the condition for stability of the steady motion becomes

$$c^2\Omega^2>4aWl. \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Also if $a>c$ there cannot be any real whirling speed; if however $a<c$ there will be such a speed, given by

$$\Omega^2=Wl/(c-a), \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and it may be shown that this will be above the minimum speed for exponential stability as given by (20).

7. THE INVERTED TOP

If the point of the peg is freely jointed, so that the top can spin suspended in an inverted position, then the preceding analysis is applicable to such cases provided

we change the sign of W. Thus we shall have for the steady motion controlled by the unbalance

$$\Theta_0 = -\frac{mld\,\Omega^2 \sin \varepsilon_d}{(b-c)\,\Omega^2 - Wl}, \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$\Phi_0 = -\frac{mld\,\Omega^2 \cos \varepsilon_d}{(a-c)\,\Omega^2 - Wl} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

If a, b, are both greater than c, there will be two real whirling speeds, namely

$$\Omega_1^2 = Wl/(b-c), \quad \Omega_2^2 = Wl/(a-c). \quad . \quad . \quad . \quad (24)$$

Clearly if in this case $a > b$ then $\Omega_2^2 < \Omega_1^2$. For the case $a > c > b$, there will be only one real whirling speed, namely that given by Ω_2^2 above.

For the inverted symmetrical top the disturbed steady motion will be exponentially stable at all speeds; also there cannot be more than one real whirling speed and then only provided that a is greater than c.

REFERENCES

1. ROUTH, E. J. (1905). Motion of a Top, Small Oscillations. *Advanced Rigid Dynamics* (6th Edition, Chap. 5, § 214, p. 160). Macmillan, 1905.

2. MORRIS, J. Unsymmetrical Load of Appreciable Moments of Inertia. *The Escalator Method in Engineering Vibration Problems* (Chap. 9, § 5, p. 99). Chapman & Hall, 1947.

THE POSSIBILITY OF THE DETERMINATION OF RATE OF CLIMB FROM ACCELERATION MEASUREMENTS IN LEVEL FLIGHT

by

E. C. PIKE, M.A., A.F.R.Ae.S.

(*Member of the Technical Staff of the Royal Aeronautical Society*)

SUMMARY

Performance measurements for turbo-jet and turbo-prop aircraft are most conveniently reduced to standard conditions by the use of graphical methods. The case of rate of climb presents difficulties if readings are taken in the normal way during continuous climbs, because of the variation of weight, temperature, pressure and optimum climbing speed with height.

Since rate of climb is proportional to the product of speed and acceleration in level flight, these quantities could be measured and used to determine the rate of climb at any height.

The advantages of the method are listed and the technique required of the pilot examined. An automatic observer, including a longitudinal accelerometer, reading, in some cases, to within 0.001g, is necessary.

The possibilities of error have been examined and have not been found to be serious.

An example is given for a typical turbo-jet aircraft of the procedure in the determination of rate of climb at any forward speed for one value of $N/\sqrt{\theta}$ and of $W/\tilde{\omega}$. To determine maximum rate of climb alone, fewer readings are sufficient. "Non-dimensional" curves of $v_c/\sqrt{\theta}$ against $N/\sqrt{\theta}$ and $W/\tilde{\omega}$ can be plotted *directly* from a series of such runs at different heights and different values of N.

1. INTRODUCTION

For the performance reduction of turbo-jet and turbo-prop aircraft, graphical

Paper received June 1948.

[The Aeronautical Quarterly, Vol. II, May 1950]

methods using the "non-dimensional" quantities $V/\sqrt{\theta}$, $N/\sqrt{\theta}$, $W/\tilde{\omega}$, T_4/θ^*, $v_c/\sqrt{\theta}$ etc. are the most convenient[1, 2].

The level speed case is fairly simple because

$$\left.\begin{aligned} V/\sqrt{\theta} &= f_1(N/\sqrt{\theta},\ W/\tilde{\omega}) \text{ for turbo-jets,} \\ &= \phi_1(N/\sqrt{\theta},\ W/\tilde{\omega},\ T_4/\theta) \text{ for turbo-props,} \end{aligned}\right\} \quad . \quad . \quad (1)$$

and during a level speed run at constant indicated height $W/\tilde{\omega}$, $N/\sqrt{\theta}$ and T_4/θ are approximately constant. Therefore for any fixed value of $W/\tilde{\omega}$, measured values of $V/\sqrt{\theta}$ can be plotted against measured values of $N/\sqrt{\theta}$ and T_4/θ (for turbo-props) or $N/\sqrt{\theta}$ alone (for turbo-jets). The value of $V/\sqrt{\theta}$ under the required standard conditions can then be read directly from the faired curves through these points.

The reduction of rate of climb results involves two difficulties, both of which are accentuated in the turbo-prop case.

First, since rate of climb is a function of forward speed the additional variable $V/\sqrt{\theta}\,(=V_1/\sqrt{\tilde{\omega}})$ is introduced into the equations, *i.e.*:—

$$\left.\begin{aligned} v_c/\sqrt{\theta} &= f_2(N/\sqrt{\theta},\ W/\tilde{\omega},\ V_1/\sqrt{\tilde{\omega}}) \text{ for turbo-jets,} \\ &= \phi_2(N/\sqrt{\theta},\ W/\tilde{\omega},\ T_4/\theta,\ V_1/\sqrt{\tilde{\omega}}) \text{ for turbo-props.} \end{aligned}\right\} \quad . \quad (2)$$

In order to simplify the flying programme and the plotting of results, in practice the indicated air speed is varied with height in a pre-determined way so that the variable $V_1/\sqrt{\tilde{\omega}}$ is related to $W/\tilde{\omega}$ and is effectively eliminated from the relations (2). With this procedure it is difficult to determine the variation of V_1 with height such that optimum rate of climb is achieved for all the values of $N/\sqrt{\theta}$, $W/\tilde{\omega}$ and T_4/θ which occur during the separate climbs. To determine the rate of climb at an arbitrary forward speed (not necessarily for maximum rate of climb) is even more difficult, and probably impossible.

Second, in the usual method of measuring rate of climb, by observing times during a continuous climb, none of the variables in (2) remains constant. This means that preliminary cross-plots of the results are necessary before $v_c/\sqrt{\theta}$ can be plotted against the other "non-dimensional" parameters. However, it is possible effectively to measure rate of climb in level flight as the product of speed and acceleration, so eliminating the difficulties.

---

* It has been assumed for simplicity of explanation that T_4/θ is the additional variable in the engine performance of turbo-props (*See* Section 3 of Ref. 3). This assumption in no way prejudices the method.

RATE OF CLIMB FROM ACCELERATION MEASUREMENTS

The equation of motion for an aircraft can be written

$$(F-D)V = \frac{d}{dt}\left(\frac{1}{2}\frac{W}{g}V^2 + Wh\right)$$

$$= W\left(\frac{V}{g}\frac{dV}{dt} + \frac{v_c}{60}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or

$$(F-D)\frac{V}{W} = \frac{v_c}{60}\left(1 + \frac{V}{g}\frac{dV}{dh}\right), \text{ for climb,} \quad . \quad . \quad . \quad (4)$$

$$= \frac{V}{g}\frac{dV}{dt}, \text{ in level flight.} \quad . \quad . \quad . \quad . \quad (5)$$

At any speed V the left hand side of equations (4) and (5) is the same whether the aircraft is climbing or not (assuming $L = W$, which is justified, particularly for turbo-jet or turbo-prop aircraft). Therefore if $\frac{V}{g}\frac{dV}{dt}$ is recorded in level flight, it can be equated to $\frac{v_c}{60}\left(1 + \frac{V}{g}\frac{dV}{dh}\right)$, which is equal to the rate of climb (in ft./sec.) at the particular height (without taking into account the acceleration term $\frac{V}{g}\frac{dV}{dh}$).

If the acceleration term is taken into account, as it should be for turbo-jet or turbo-prop aircraft, $\frac{V}{g}\frac{dV}{dh}$ is readily calculated from runs at different heights. The value of V used in determining $\frac{V}{g}\frac{dV}{dh}$ will usually be the optimum at each height (*i.e.* the speed for the greatest value of $V\frac{dV}{dt}$, or maximum rate of climb, at that height).

The possibilities of using this method in practice are discussed here.

Notation

The suffix "a" denotes ambient or test conditions, the suffix "s" denotes standard conditions.

D	total aircraft drag (lb.)
F	propulsive thrust (lb.)
f_1, f_2	functions in equations (1) and (2)
g	acceleration due to gravity (ft./sec.^{2})
h	height (ft.)

I.A.S.	indicated air speed (knots)
L	total lift (lb.)
N	engine speed (r.p.m.)
r.p.m.	revolutions per minute
T_4	absolute temperature in jet pipe (°K.)
t	time (sec.)
V	true air speed (ft./sec.)
V_{max}	maximum true air speed (ft./sec).
V_i	equivalent air speed (knots)
v_c	rate of climb (ft./min.)
W	aircraft weight (lb.)
θ	ratio of absolute temperature to standard sea level value of absolute temperature
$\tilde{\omega}$	ratio of static pressure to standard sea level value of static pressure
σ	ratio of density to standard sea level density
ϕ_1, ϕ_2	functions in equations (1) and (2)

2. ADVANTAGES OF THE METHOD

2.1. *For turbo-jets*

(*i*) Flying time can be saved by measuring rate of climb during the level speed run. If the aircraft is accelerated from about $0.5V_{max.}$ to $0.99V_{max.}$ at a constant indicated height and constant N, measurements of rate of climb and maximum level speed can be made during the same run.

(*ii*) During the run a "partial climb" is measured at the same time, *i.e.* the rate of climb at any forward speed is known, and therefore the optimum value for the particular values of $N/\sqrt{\theta}$ and $W/\tilde{\omega}$ is ensured. This avoids making separate partial climbs to determine the optimum climbing speed as in the case of continuous climbs and therefore, the uncertainty is eliminated of the use of just one climbing speed at each height for the various values of $N/\sqrt{\theta}$ and $W/\tilde{\omega}$ which occur during separate continuous climbs.

In addition, the optimum climbing speed at each height may not give the best time to height, but results obtained by the suggested method can be used to determine immediately what is the optimum variation of forward speed with height to give the best time to height.

(*iii*) During a run at constant N and constant indicated height, $W/\tilde{\omega}$ and $N/\sqrt{\theta}$ are constant. Therefore preliminary cross-plots of the results are not necessary, as they are for continuous climbs, where $W/\tilde{\omega}$ and $N/\sqrt{\theta}$ are varying continuously. This saves a considerable amount of time in the analysis and plotting of the results.

(*iv*) By the use of the method suggested, allowance can be made for the acceleration term.

(*v*) In obtaining v_c from relations (4) and (5), the true rate of climb is determined directly. During continuous climbs the rate of change of pressure height is measured and has first to be multiplied by the factor θ_s/θ_a to obtain the true rate of climb.

2.2. *For turbo-props*

The advantages claimed above for the turbo-jet case apply with even greater force to turbo-props. This is because the appearance of the third variable T_4/θ in the engine performance equations complicates the flight testing and the analysis to an enormous extent. If continuous climbs are used, it is doubtful if the climbing speed variation selected is actually the optimum for all three variables and in addition a considerable amount of preliminary cross-plotting of the results is necessary, because of the continuous variation of all the "non-dimensional" parameters.

3. PILOTING TECHNIQUE REQUIRED

It is assumed that an automatic observer is fitted, recording N, I.A.S., height, fuel contents, longitudinal acceleration and, in the case of the turbo-prop, jet pipe temperature. Outside air temperature is preferably recorded during the flight.

The pilot is required to climb to the selected height, set the engine controls to the selected value of N (and to maintain the selected T_4 for turbo-props) and then fly at constant altimeter reading during the acceleration from about $0.5V_{max.}$ to $0.99V_{max.}$ The second and third runs (for different N) would be made at slightly different altimeter readings to allow for the fuel consumed during the flight. For instance, for the turbo-jet aircraft given in the example, the first run would be at 20,000 ft., the second at 21,000 ft. and the third at 22,000 ft., so that $W/\tilde{\omega}$ is the same for each run.

4 POSSIBILITIES OF ERROR

The pilot may experience difficulty in maintaining constant indicated height during the inevitable changes of trim which will occur during the acceleration but it is not expected that the error would be serious. Other errors which may arise are:

(*i*) Constant altimeter readings may not give a constant value of $W/\tilde{\omega}$ due to fuel consumption during the run, and to position error.

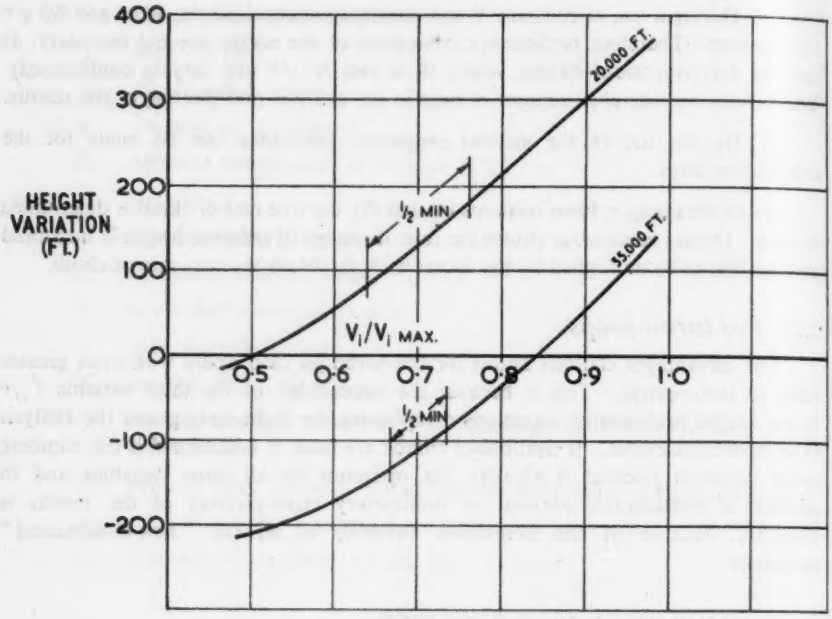

Fig. 1.
Climbing effect.

(*ii*) Increase of actual height at constant indicated height, due to position error, means that the aircraft is actually climbing very slightly.

(*iii*) The longitudinal accelerometer may not be sufficiently sensitive.

The significance of these errors is considered in what follows.

4.1. *Error due to variation of $W/\tilde{\omega}$ during the run*

Two factors affect $W/\tilde{\omega}$ during the run; the reduction in W due to fuel consumption, and the change in $\tilde{\omega}$ at constant altimeter reading due to position error. For aircraft with pitot heads having an altimeter correction which increases positively with speed, the effects of altimeter error and weight reduction act in opposite senses. Calculations for the typical turbo-jet aircraft of the example at 20,000 ft. and 35,000 ft. show that between 50 per cent. and 90 per cent. of $V_{max.}$, which covers the climb region, a constant value of altimeter reading gives a value of $W/\tilde{\omega}$ which is constant within 0.6 per cent. Since nearly all jet aircraft have leading edge pitot heads, for which the altimeter correction increases positively with speed, the assumption of constant altimeter reading is justified.

4.2. *Error due to slight climbing effect during run*

If constant indicated height is maintained during the run, and the altimeter correction increases positively with speed, the aircraft actually gains height. A slight error in the deduced rate of climb will result. Fig. 1 shows approximately the change of height for the typical turbo-jet aircraft used in the example. The error is seldom likely to be more than 6 per cent. and a correction can easily be made if necessary.

4.3. *Error due to sensitivity of the accelerometer*

As the example shows, a rate of climb of 3,000 ft./min. at 220 knots corresponds to about 0.1g, so that for lower rates of climb the sensitivity of the accelerometer would have to be of the order of 0.001g. This accuracy has already been obtained with an improvised instrument, so that there should be no danger of error from this source, particularly as the development of accelerometers proceeds.

5. EXAMPLE

During a run at 20,000 ft. at 14,000 r.p.m. the following figures might be observed for a typical turbo-jet aircraft. The instrument readings have been corrected.

$$N/\sqrt{\theta_a} = \frac{13{,}940}{0.921} = 15{,}140$$ (from corrected value of r.p.m. and temperature at time of test)

$$W/\tilde{\omega}_a = \frac{24{,}500}{0.450} = 54{,}400$$ (from fuel contents and altimeter reading)

also $\sigma_a = \tilde{\omega}_a/\theta_a = 0.53$.

V_1: knots	158	178	197	217	237	257	276	287	329
longitudinal acceleration: g	0.118	0.110	0.107	0.100	0.0928	0.0834	0.0652	0.0471	0

and therefore, using $v_c = \text{acceleration} \times \dfrac{V_1 \times 1.689 \times 60}{\sqrt{(0.53)}}$,

v_c : ft./min.	2590	2720	2940	3020	3060	2980	2500	1880	0

These results are plotted in Fig. 2 and give a complete partial climb curve for $N/\sqrt{\theta} = 15{,}140$, $W/\tilde{\omega} = 54{,}400$. To obtain only the *maximum* rate of climb it would not be necessary to plot as many points.

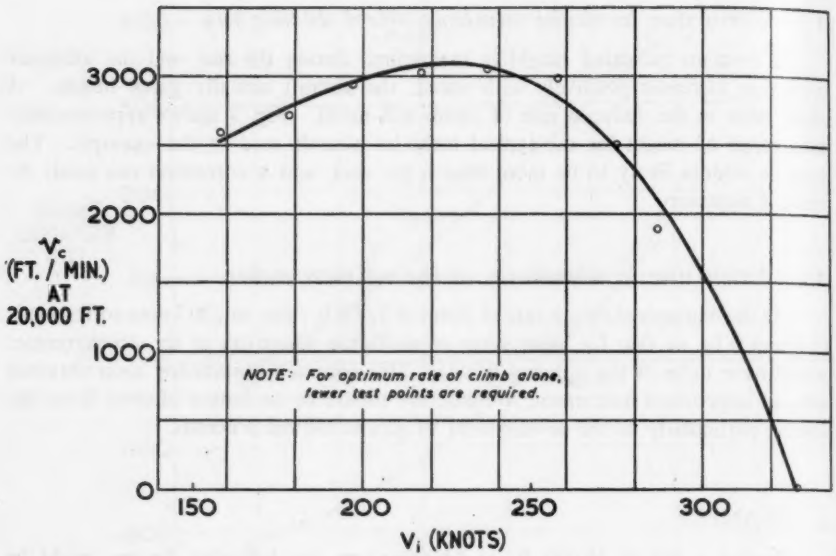

Fig. 2.
Example of results.

For maximum rate of climb, in this case

$$v_c/\sqrt{\theta} = \frac{3080}{0.921} = 3340.$$

The complete series of runs will give a "non-dimensional" set of curves from which the maximum rate of climb under standard conditions at any height and for any value of N can be obtained, and allowance can be made for acceleration during a continuous climb.

A similar set of curves can be drawn giving the rate of climb at any arbitrary forward speed.

The advantage of the fact that the "non-dimensional" parameters are constant during the run is apparent during the plotting of the complete results for various heights and values of N.

6. REFERENCES

1. Royal Aeronautical Society's Performance Data Sheets.
2. SMITH, F. (1943). Note on various Methods of Performance Reduction for Jet-Propelled Aircraft (with addendum). R.A.E. Tech. Note. No. Aero. 1348 and 1348*a*, 1943.
3. PIKE, E. C. (1949). Note on Propeller Turbine Reduction Methods. *The Aeronautical Quarterly*, Vol. I, May 1949.

A NOTE ON SUBSONIC AEROFOIL THEORY

by

JOHN W. MILES, Ph.D.

(Department of Engineering, University of California, Los Angeles)

SUMMARY

A linearised theory for the subsonic, lifting surface problem is formulated in terms of Fourier integral solutions to Laplace's equation. The symmetric and anti-symmetric problems of the first kind are solved explicitly, while the problems of the second kind depend on the solution of dual integral equations. The anti-symmetric problem of the second kind is cast in a variational form, from which certain well-known theorems may be deduced.

1. INTRODUCTION

The following development of linearised, subsonic aerofoil theory is derived from a recent analysis given for the supersonic problem[1] which in turn was motivated by Schwinger's variational approach to diffraction problems[2, 3]. While the majority of the end results to be obtained are not new, their formulation, in certain respects, is rather novel, and the analysis is felt to be of interest *per se*.

Earlier formulations of linearised, lifting surface theories are discussed by von Kármán[4], and rather general approaches have been given more recently by Kussner[5] and Reissner[6]. A Fourier transform approach of lifting line theory has been discussed by von Kármán[7] and Fuchs[8].

Notation

a	x co-ordinate of axis of rotation in pitch
b	wing span
c	reference value of chord
C_D	drag coefficient
C_L	lift coefficient
C_l	rolling moment coefficient
C_M	pitching moment coefficient

Paper received April 1949.

[The Aeronautical Quarterly, Vol. II, May 1950]

G	*see* equation (7)
g	*see* equation (9)
G^*	*see* equation (12)
g^*	*see* equation (13)
$\mathscr{I}$	"the imaginary part of"
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors along x, y, z
K_0	MacDonald's function of order zero[12]
K_1	MacDonald's function of order one[12]
p	perturbation pressure
$\mathbf{q}$	vector perturbation velocity
R	integration operator. *See* equation (8)
T	transform operator. *See* equation (4*a*)
$\overline{T}$	inverse of T
u, v, w	x, y, z components of perturbation velocity
U	Fourier transform of u. *See* equation (4)
V	free stream velocity
x, y, z	Cartesian co-ordinates (Fig. 1)
α	wing incidence ($\alpha = -w$ in σ)
$\partial(\mu)$	Dirac delta function; $\int_{-\infty}^{\infty} \delta(\mu) = 1$; $\delta(\mu) = 0$, $\mu \neq 0$.
η	y co-ordinate of integration
μ, ν	spectrum co-ordinates
ξ	x co-ordinate of integration
ρ	air mass density
σ	projeetion of aerofoil on $z=0$
τ	remainder of plane $z=0$
ϕ	velocity potential

2. STATEMENT OF PROBLEM

A thin wing is located near the $xy\,(z=0)$ plane, its projection there being denoted by σ (the remainder of the plane being denoted by τ). The theory is linearised in the sense that the boundary conditions are applied in σ rather than at the aerofoil, and all perturbation velocities are assumed small compared to the free

A NOTE ON SUBSONIC AEROFOIL THEORY

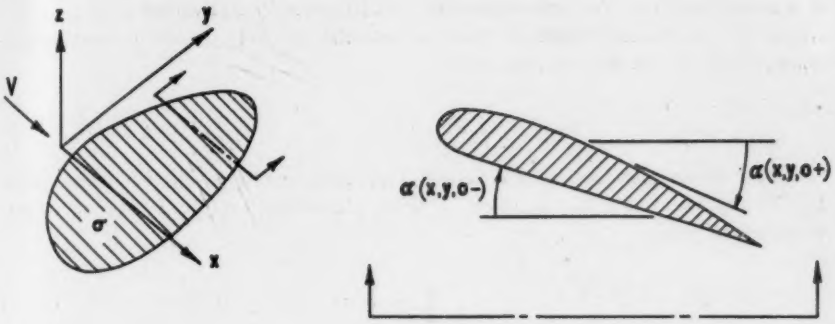

Fig. 1.
Plan and section views of aerofoil.

stream velocity, V, which is in the positive x direction. The perturbation velocity may be written

$$\mathbf{q}=(\mathbf{i}u+\mathbf{j}v+\mathbf{k}w)\,V=V\nabla\phi \quad . \quad . \quad . \quad . \quad (1)$$

$$\nabla^2\phi=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the (linearised) perturbation (relative to free stream) pressure is given by

$$p=-\rho V^2u \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where ρ is the mass density. It is required to find $\phi\,(x, y, z)$ from a knowledge of the boundary conditions in the plane $z=0$ and at infinity.

The boundary conditions require that the flow be tangent to the aerofoil and that the pressure be continuous except across the aerofoil. The top and bottom surfaces of the aerofoil will generally have different slopes, but the linearisation permits separate consideration of two problems consisting of the symmetrical and anti-symmetrical distributions of the incidence (measured in a section of constant y), corresponding to the thickness and camber (and twist) distributions, respectively. Hence, it suffices to consider only the half space $z>0$, the boundary conditions being applied to $u\,(x, y, 0+)$ or $w\,(x, y, 0+)$ (the case whose $v\,(x, y, 0+)$ is specified is not generally of technical interest).

In addition to the explicit boundary conditions in the plane $z=0$, it is required that the solutions be chosen such that ϕ will vanish far upstream, reduce to a

two-dimensional flow far downstream (the Trefftz plane)[9] and vanish as $y, z \to \infty$. Moreover, the Kutta condition must be satisfied, *i.e.* $u(x, y, 0+)$ must vanish continuously at the trailing edge of σ.

3. SOLUTION

Let lower and upper case letters represent functions of the co-ordinates (x, y) and their double, complex, Fourier transforms in the (μ, ν) spectrum, respectively, *e.g.*

$$u(x, y)=T\{U(\mu, \nu)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\mu \int_{-\infty}^{\infty} d\nu\, U(\mu, \nu)\exp\,[i(\mu x+\nu y)]\ . \tag{4a}$$

$$U(\mu, \nu)=T\{u(x, y)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta\, u(\xi, \eta)\exp\,[-i(\mu\xi+\nu\eta)]\ . \tag{4b}$$

Then a solution to Laplace's equation (2) which exhibits the desired behaviour at infinity (in the half space $z>0$) and whose x derivative reduces to $u(x, y)$ at $z=0+$ may be established as

$$\phi(x, y, z)=T\{[(i\mu)^{-1}+\pi\delta(\mu)]\exp[-(\mu^2+\nu^2)^{\frac{1}{2}}z]\,U(\mu, \nu)\}\ . \tag{5}$$

The branch cuts in the complex ν plane are taken from $\pm i|\mu|$ to $\pm i\infty$, respectively, and the path of integration in the complex ν plane may be taken anywhere in the strip $|\mathscr{I}(\nu)|<|\mathscr{I}(\mu)|$, provided $z\geqslant 0$, as assumed. Proper behaviour for large values of y is guaranteed by giving $\mathscr{I}(\nu)$ the same sign as y, with the real axis admissible for all y. The path in the μ plane is taken along $\mathscr{I}(\mu)\leqslant 0$, and the contour may be closed in the upper or lower half of the μ plane for large positive or negative x, respectively, giving a vanishing solution far upstream and a potential solution (the contribution of the pole) far downstream. If the path is taken along the real axis in the μ plane, the indentation *under* the pole $\mu=0$ yields equal and opposite contributions upstream and downstream, which are then matched by the contribution from the term $\pi\delta(\mu)$[10] in such a way as to give the correct solutions both upstream and downstream. These choices determine essentially the arbitrary function of (x, y) introduced when $u(x, y, z)$ is integrated with respect to x to determine $\phi(x, y, z)$. The Kutta condition does not enter the solution explicitly at this point since it is implicit in the specification of $u(x, y)$.

The upwash corresponding to equation (5) may be written,

$$w(x, y)=\phi_z(x, y, 0+)=-T\{G(\mu, \nu)U(\mu, \nu)\exp[-(\mu^2+\nu^2)^{\frac{1}{2}}(0+)]\}\ . \tag{6}$$

$$G(\mu, \nu)=(i\mu)^{-1}(\mu^2+\nu^2)^{\frac{1}{2}}+\pi\delta(\mu)|\nu|\ . \tag{7}$$

The requirement $z=0+$ in equation (6) guarantees absolute convergence of the integrals in the limits $\mathscr{J}(\mu)=\mathscr{J}(\nu)=0$, but in the following the results will be formally interpreted for $z=0$ (*cf.* footnote †). Applying the Faltung theorem[11] to equation (6) yields

$$w(x, y)=-(2\pi)^{-1} R\{g(x-\xi, y-\eta)\, u(\xi, \eta)\}$$

$$=-(2\pi)^{-1}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta\, g(x-\xi, y-\eta)\, u(\xi, \eta) \quad . \quad . \quad (8)$$

where R is an operator denoting double integration over all (ξ, η), and $g(x, y)$ is the inverse transform of $G(\mu, \nu)$. Carrying out the integrations required for the

$$g(x, y)=-(i\pi\,|\,y\,|)^{-1}\int_{-\infty}^{\infty} d\mu\, K_1(\mu\,|\,y\,|)\exp(i\mu x)-y^{-2} \quad . \quad . \quad (9a)$$

$$=-2(\pi\,|\,y\,|)^{-1}\int_{0}^{\infty} d\mu\, K_1(\mu\,|\,y\,|)\sin(\mu x)-y^{-2} \quad . \quad . \quad (9b)$$

$$=-y^{-2}[1+x(x^2+y^2)^{-\frac{1}{2}}] \quad . \quad . \quad . \quad . \quad . \quad . \quad (9c)$$

where the path of integration for equation (9*a*) is indented under the pole $\mu=0$, and K_1 is a modified Bessel function of the second kind in the notation of Watson[12].

The result obtained by substituting equation (9*c*) in equation (8) evidently has a direct interpretation in terms of a source distribution over $z=0+$, and, after integration by parts to give integrals possessing Cauchy principal values†, it is in agreement with results obtained by rather more direct approaches.

4. INVERSION OF INTEGRAL EQUATION

Equations (6) and (8) represent a direct, closed form solution for $w(x, y)$ if $u(x, y)$ is known everywhere for $z=0+$. Conversely, if $w(x, y)$ is known these results are integral equations of the first kind for $U(\mu, \nu)$ and $u(x, y)$, respectively. The integral equation may be inverted by taking the inverse transform‡ of

---

† If z is kept finite (>0), it is found that

$$g(x, y)=-\frac{\partial}{\partial z}\left\{z(z^2+y^2)^{-1}[1+x(x^2+y^2+z^2)^{-1/2}]\right\}$$

and no difficulties occur at $z=0$, after integration by parts.

‡In taking the reciprocal of $G(\mu, \nu)$, *cf.* equation (7), the delta function drops out since, in any event, $G^*(\mu, \nu)$ has a first order zero at $\mu=0$. *See*, for example, Ref. 10, p. 21.

equation (6) and applying the Faltung theorem, namely

$$u(x, y) = -T\{G^*(\mu, \nu) W(\mu, \nu)\} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$u(x, y) = -(2\pi)^{-1} T\{g^*(x-\xi, y-\eta) w(\xi, \eta)\} \quad . \quad . \quad (11)$$

$$G^*(\mu, \nu) = G^{-1}(\mu, \nu) = i\mu(\mu^2+\nu^2)^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Taking the inverse transform of equation (12) yields

$$g^*(x, y) = (i/\pi) \int_{-\infty}^{\infty} \mu\, d\mu\, K_0(\mu\,|\,y\,|) \exp(i\mu x) \quad . \quad . \quad . \quad (13a)$$

$$= -(2/\pi) \int_{0}^{\infty} \mu\, d\mu\, K_0(\mu\,|\,y\,|) \sin(\mu x) \quad . \quad . \quad (13b)$$

$$= -x(x^2+y^2)^{-3/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13c)$$

Here again, the results are formal, convergence being obtained through an appropriate choice of paths in the transform planes and interpretation at $z=0+$, followed by integration by parts in (x, y).

5. PROBLEMS OF FIRST AND SECOND KIND

In a problem of the first kind either u or w is known over the entire plane $z=0+$, and the results of the last two sections therefore furnish exact solutions. In a problem of the second kind u and w are specified over different (but compatible) portions of the plane, and it is necessary to solve a "dual" integral equation†.

Referring to the symmetric (s) and anti-symmetric (a) problems previously mentioned, w_s and u_a vanish over that portion of the plane (τ) not occupied by the aerofoil (σ); hence, the determinations of either u_s or w_a from a knowledge of w_s or u_a, respectively, in σ leads to a problem of the first kind, while the determination of u_a or w_s from w_a or u_s, respectively, in σ leads to a problem of the second kind. The only exact solutions known for cases of interest in the latter category are the solutions for the circular[16] and elliptic[17] plan forms.

6. UNIQUENESS

While no adequate treatment of dual integral equations of the first kind appears to exist in the literature, they may be expected to exhibit characteristics similar to

† *See* Ref. 11, pp. 334-342, for discussion of this type of integral equation.

other Fredholm integral equations arising in mathematical physics. (*See*, for example, Ref. 15.) In particular, eigensolutions may be expected for any given plan form, and these will generally possess singularities at the plan form edges. Various singularities are possible, but only singularities which are integrable, in the sense that they yield finite forces on the wing, can be admitted in the present problem, namely logarithmic and $(-\frac{1}{2})$ power singularities, in the symmetric and anti-symmetric problems, respectively[16]. Now the particular solution to the integral equation may also be expected to exhibit a $(-\frac{1}{2})$ power singularity (in the anti-symmetric problem) at the plan form edges, so that the eigensolution may be chosen in such a way as just to cancel this singularity along the trailing edge, thus satisfying the Kutta condition and establishing the solution as unique.† The singularity along the leading edge remains and indeed is required (on the assumption of an ideal fluid) to eliminate drag (other than induced).

The question of uniqueness is a recondite one, both mathematically and physically (no physical answer to the question can be given without considering the effects of viscosity in some detail), and it cannot be said that the question has been answered in any completely satisfactory way. It is, therefore, of some interest to remark that the same situation exists in other fields, *e.g.* diffraction[17], and has often been ignored. An interesting, if laconic, discussion of the supersonic case has been given recently by Hayes[18].

7. VARIATIONAL FORMULATION

A variational formulation of the dual integral equation of the first kind has been given by Schwinger[1, 2]. Such a formulation of the present problem leads to a variational expression for the drag and is of interest only for the lifting surface problem of the second kind (angle of attack, *i.e.* downwash, distribution prescribed over aerofoil), since it is known, *a priori*, that the drag for a symmetrical aerofoil must vanish (ideal, irrotational flow).

The drag is obtained by integrating the product of pressure, *cf.* equation (3), and downwash ($w_a(x, y) = -\alpha_a(x, y)$) over the aerofoil ($\sigma$). The result is‡

$$C_D = (\rho V^2 \sigma/2)^{-1} R \{ [p(x, y, 0-) - p(x, y, 0+)] \alpha_a(x, y) \}$$

$$= (4/\sigma) R \{ u_a(x, y) \alpha_a(x, y) \} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

†It is also of interest to note that the discontinuity across the vortex sheet behind the wing is uniquely determined by the Kutta condition, or by any other uniqueness condition which might be formulated.

‡The matter of leading edge thrust (*see*, for example, Ref. 4, pp. 26-29) is passed over in silence here, except to state that the end results require no correction for such a thrust. It is not necessary to restrict the integration to σ, since u_a vanishes off σ.

It follows from Parseval's theorem† that equation (14) may also be written

$$C_D=(4/\sigma)R\{U(\mu,\nu)\overline{A(\mu,\nu)}\}$$

$$=(4/\sigma)R\{\overline{U(\mu,\nu)}A(\mu,\nu)\} \quad . \quad . \quad . \quad . \quad (15)$$

where the bar denotes the complex conjugate. Alternative forms are given by substituting $\alpha(x, y)$ or $A(\mu, \nu)$ from the integral equations (6) or (8), respectively, to obtain

$$C_D=(2/\pi\sigma)R^2\{u(x, y)g(x-\xi, y-\eta)u(\xi, \eta)\} \quad . \quad . \quad . \quad . \quad (16)$$

$$C_D=(4/\sigma)R\{U(\mu, \nu)G(\mu, \nu)\overline{U(\mu, \nu)}\} \quad . \quad . \quad . \quad . \quad (17)$$

where R^2 indicates integration over both (x, y) and (ξ, η). If equations (16) and (17) are divided by the squares of equations (14) and (15), respectively, the results are

$$C_D^{-1}=(\sigma/8\pi)[R\{u(x, y)\alpha(x, y)\}]^{-2}\times R^2\{u(x, y)g(x-\xi, y-\eta)u(\xi, \eta)\} \quad (18)$$

$$C_D^{-1}=(\sigma/4)[R\{U(\mu, \nu)\overline{A(\mu, \nu)}\}]^{-2}\times R\{G(\mu, \nu)|U(\mu, \nu)|^2\} \quad . \quad . \quad (19)$$

Equations (18) and (19) are variational in the sense that they are absolute minima (not merely stationary) with respect to arbitrary (not merely first order) variations of u or U about the exact solutions to the integral equations (6) and (8), respectively. A proof of this assertion, due to Schwinger, has been given elsewhere‡, it being specifically remarked that the present kernel is symmetric and positive definite.

To prove the assertion that the kernel is positive definite, it may be observed that, since $u(x, y)$ is real, its transform satisfies the relation

$$U(-\mu, -\nu)=\overline{U(\mu, \nu)} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

whence $U(\mu, \nu)\overline{U(\mu, \nu)}$ is an even function of μ and ν, and only that part of $G(\mu, \nu)$ which is even in μ, *i.e.* $\pi\delta(\mu)|\nu|$, *cf.* equation (7), contributes to the result. In particular, equation (17) reduces to

$$C_D=\left(\frac{8\pi}{\sigma}\right)\int_0^\infty \nu d\nu\,|U(0, \nu)|^2 \quad . \quad . \quad . \quad . \quad (21)$$

---

†Ref. 11, p. 51 (2.1.8).

‡*See* Appendix IV of Ref. 19, the proof being for a vector integral equation with dyadic kernel.

as obtained (essentially) by von Kármán[7]. This result corresponds to the calculation of the induced drag in the Trefftz plane ($x=\infty$ or $\mu=0$). It may be remarked further that

$$U(0,\ \nu)=(2\pi)^{-1}\int_{-\infty}^{\infty} d\eta \exp(-i\nu\eta)\int_{-\infty}^{\infty} d\xi\, u(\xi,\eta) \quad . \quad . \quad . \quad (22)$$

and is therefore essentially the Fourier transform of the circulation about the "lifting line." The result that the induced drag is independent of chordwise distribution of pressure (for fixed circulation) is of course the well-known Munk stagger theorem. (Ref. 4, p. 132.)

The variational equivalent of equation (21) is deduced from equation (19) as

$$C_D^{-1}=\left(\frac{\pi\sigma}{2}\right)\int_0^{\infty} \nu d\nu \left|\frac{U(0,\ \nu)}{R\{U(\mu,\ \nu)\overline{A(\mu,\ \nu)}\}}\right|^2 \quad . \quad . \quad . \quad (23)$$

The result that the reciprocal of C_D is a minimum, so that C_D is a maximum, with respect to variations about the true solution for the circulation is not new, for the usual (Fourier series) method of solving the Prandtl, lifting line, integral equation generally exhibits C_D in a positive definite, quadratic (in the expansion coefficients) form. *See*, for example, the work of Hildebrand[20].

8. LIFT AND MOMENT COEFFICIENTS

To complete the specification of the forces acting on the wing, the lift and moment coefficients will be expressed in terms of the transforms.

The lift coefficient is defined by

$$C_L=(\rho V^2\sigma/2)^{-1}R\{p(x,\ y,\ 0-)-p(x,\ y,\ 0+)\}$$
$$=\left(\frac{4}{\sigma}\right)R\{u_a(x,\ y)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Similarly, the pitching moment about $x=a$ and the rolling moment about the line $y=0$ are referred to the coefficients

$$C_M=\left(\frac{4}{\sigma c}\right)R\{(a-x)u_a(x,\ y)\} \quad . \quad . \quad . \quad . \quad (25)$$

$$C_l=-\left(\frac{4}{\sigma b}\right)R\{yu_a(x,\ y)\} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

where c is a reference wing chord, and b is the wing span. These integrals may be expressed in terms of $U(\mu, \nu)$, just as in the case of the circulation, *cf.* equation (22), since u_a vanishes off the wing. The results are

$$C_L = \left(\frac{8\pi}{\sigma}\right) U(0, 0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$C_M = \left(\frac{8\pi}{\sigma}\right)\left\{\left(\frac{a}{c}\right) U(0, 0) - \left(\frac{i}{c}\right)\left[\frac{\partial U}{\partial \mu}(\mu, 0)\right]_{\mu=0}\right\} \quad . \quad . \quad (28)$$

$$C_l = -\left(\frac{8\pi i}{\sigma b}\right)\left[\frac{\partial U}{\partial \nu}(0, \nu)\right]_{\nu=0} \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

REFERENCES

1. MILES, J. W. (1948). On Linearized Supersonic Airfoil Theory. North American Aviation Company, Aerophysics Lab. Report AL-801, 1948.
2. SCHWINGER, J. S. (1944). Seminar on the Theory of Guided Waves. Radiation Laboratory, Massachusetts Institute of Technology, Cambridge, Mass., 1944.
3. LEVINE, H., and SCHWINGER, J. (1948). On the Theory of Diffraction by an Aperture in an Infinite Plane Screen, I. *Physical Review* 74, 958-974, 1948.
4. DURAND, W. F. (1942). *Aerodynamic Theory*, Part II, Div. E. Durand Reprinting Committee, Pasadena, 1942.
5. KUSSNER, H. G. (1940). Allgemeine Tragflachentheorie. *Luftfahrtforschung*, 17, 370-378, 1940; translated in N.A.C.A. Technical Memorandum 979, 1941.
6. REISSNER, E. (1944). On the General Theory of Thin Airfoils for Non-Uniform Motion. N.A.C.A. Technical Note 946, 1944.
7. VON KARMAN, TH. (1935). Neue Darstellung der Tragflügeltheorie. *Zeitschrift für Angewandte Mathematik und Mechanik* 15, 56-61, 1935.
8. FUCHS, R. (1939). Neue Behandlung der Tragflügeltheorie. *Ingenuieur Archiv*. 10, 48-63, 1939.
9. TREFFTZ, E. (1921). Zur Prandtlschen Tragflächentheorie. *Math. Annal.* 82, 306-319, 1921.
10. DIRAC, P. (1935). *Quantum Mechanics*, Oxford, pp. 21 *et seq.*, 1935,
11. TITCHMARSH, E. C. (1937). *Theory of Fourier Integrals*, Oxford, p. 50 (2.1.2), 1937.
12. WATSON, G. N. (1947). *Bessel Functions*, MacMillan & Co., New York, 1947.
13. KINNER, W. (1937). Die kriesförmige Tragfläche auf potential-theoretischer Grundlage. *Ingenieur Archiv*. 8, 47-80, 1937.
14. KRIENES, K. (1940). Die elliptische Tragfläche auf potential-theoretischer Grundlage. *Zeitschrift für Angewandte Mathematik und Mechanik* 20, 65-68, 1940; translated in N.A.C.A. Technical Memorandum 971, 1941.
15. COURANT, R., and HILBERT, D. (1937). *Methoden der Mathematischen Physik*, Julius Springer, Berlin, Vol. II, p. 269 *et seq.*, 1937.

16. LAGERSTROM, P. A. (1948). Linearized Supersonic Theory of Conical Wings. N.A.C.A. Technical Note 1685, pp. 12-14, 71, 1948.

17. BOUWKAMP, C. J. (1947). (In reviewing E. T. Copson's *An Integral Equation Method of Solving Plane Diffraction Problems*), Mathematical Reviews 8, 180, 1947.

18. HAYES, W. D. (1947). Linearized Supersonic Flow. California Institute of Technology. Ph. D. Thesis 1947; also distributed as North American Aviation Company AL-222, pp. 66-75, 1947.

19. MILES, J. W. (1946). The Equivalent Circuit for a Plane Discontinuity in a Cylindrical Wave Guide. *Proceedings of the Institute of Radio Engineers* 10, 728-742, 1946.

20. HILDEBRAND, F. B. (1944). A Least Squares Procedure for the Solution of the Lifting Line Integral Equation. N.A.C.A. Technical Note 925, 1944.

THEORY OF AN OSCILLATING SUPERSONIC AEROFOIL

by

GEOFFREY L. SEWELL, B.A.

(King's College, London)

In the previous linearised theories of steady supersonic flow past a slender oscillating aerofoil, the shock line at the nose has been assumed to be a straight, stationary one, so that no vorticity is produced on passing through it. In the present treatment, the possibility that the shock may be both curved and moving is taken into account, but the result is exactly the same as that obtained by the previous treatments (*e.g.* Temple and Jahn[1]).

Notation

$\mathbf{u}_0$, p_0, ρ_0, a_0, μ_0	constant values of flow velocity, pressure, density, speed of sound and Mach angle in the free stream
Ox, Oy, Oz	rectangular Cartesian fixed right-handed axes. O mean position of nose, Ox direction of free stream, Oz direction of axis of rotation of aerofoil.
t	time from a given fixed instant
$\mathbf{u}$, p, ρ	perturbation values of velocity, pressure and density respectively after passing through the shock
$v_0(t)$	transverse velocity of nose in direction Oy at time t
$\theta(t)$	inclination of aerofoil to Ox at time t
$\omega(t)$	angular velocity of aerofoil about Oz at time $t\,(=\dot{\theta}(t))$
$\mathbf{k}$	unit vector in direction Oz
H, α, f	introduced and defined in context
S	shock curve
P	a point on S
ψ	inclination of tangent at P to Ox at time t
L	the line $y=x\tan\mu$
l_0, m_0	flow velocity components immediately behind P normal and tangential to the shock
l, m	corresponding quantities immediately in front of P

Paper received May 1949.

$\epsilon, \phi(x, y, t), d(x, t)$ introduced and defined in context

γ ratio of specific heat at constant pressure to specific heat at constant volume

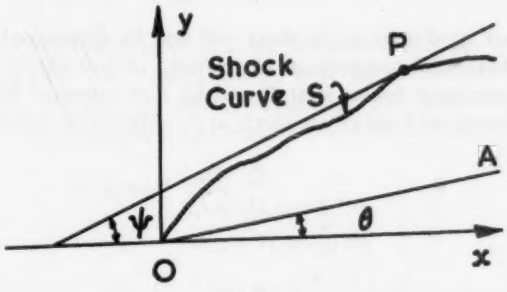

Fig. 1.

Consider first the case of a flat aerofoil OA in a stream of constant speed u_0, whose fixed direction will be taken to be that of the x-axis. Let Oy be the line perpendicular to Ox in the plane of oscillation, and let Oz complete a right-handed set of axes Ox, Oy, Oz. The aerofoil is oscillating with angular velocity $\omega(t)(=\dot{\theta})$, while the nose, O, is vibrating along Oy with velocity $v_0(t)$. Let $\mathbf{u}_0$, p_0, ρ_0 be the velocity, pressure and density respectively in the initial uniform stream, and let $\mathbf{u}_0+\mathbf{u}$, p_0+p, $\rho_0+\rho$ denote the corresponding qantities at $(x, y, 0)$, a point in front of the shock. Moreover, since p, ρ, $\mathbf{u}$, v_0, ω, θ are all small quantities whose squares we shall neglect, the equations of continuity, motion and energy become respectively

$$\frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial t}+u_0\frac{\partial \rho}{\partial x}\right)+\operatorname{div}\ \mathbf{u}=0, \qquad (1)$$

$$\frac{1}{\rho_0}\operatorname{grad} p+\frac{\partial \mathbf{u}}{\partial t}+u_0\frac{\partial \mathbf{u}}{\partial x}=0, \qquad (2)$$

$$\frac{\partial p}{\partial t}+u_0\frac{\partial p}{\partial x}=a_0^2\left(\frac{\partial \rho}{\partial t}+u_0\frac{\partial \rho}{\partial x}\right), \qquad (3)$$

where a_0 is the speed of sound in the initial uniform stream. Carrier[2] has shown that the solution of these equations may be written in the form

$$\left.\begin{aligned}\mathbf{u}&=\operatorname{grad} H+\operatorname{curl}(\alpha\mathbf{k}),\\ p&=-\rho_0\left(\frac{\partial H}{\partial t}+u_0\frac{\partial H}{\partial x}\right),\\ \rho&=-\frac{\rho_0}{a_0^2}\left(\frac{\partial H}{\partial t}+u_0\frac{\partial H}{\partial x}\right)+f(x-u_0t,\ y),\end{aligned}\right\} \qquad (A)$$

where **k** is the unit vector in the direction Oz, $\alpha=\alpha(x-u_0t,\ y)$ and H satisfies the equation

$$a_0^2\nabla^2 H=\left(\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}\right)^2 H. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The boundary conditions at the shock will now be determined. Let l_0, m_0 be the normal and tangential components, respectively, of the velocity of flow relative to the shock immediately behind it at P; and let l, m represent the corresponding quantities immediately in front of the shock at P. The shock equations are

$$m=m_0,$$
$$(\rho+\rho_0)\,l=\rho_0 l_0,$$
$$p+(\rho+\rho_0)\,l^2=\rho_0 l_0^2,$$
$$\frac{2\gamma}{\gamma-1}\left(\frac{p+p_0}{\rho+\rho_0}\right)+l^2=\frac{2\gamma}{\gamma-1}\frac{p_0}{\rho_0}+l_0^2,$$

γ being the ratio of the specific heats of the gas. The solution of these equations is

$$m=m_0,$$
$$l-l_0=\frac{2}{\gamma+1}\frac{a_0^2-l_0^2}{l_0},$$
$$p=\frac{2}{\gamma+1}\rho_0(l_0^2-a_0^2),$$
$$\rho=\frac{2}{\gamma+1}\rho_0\frac{(l_0^2-a_0^2)}{ll_0}.$$

Since the shock is weak, l_0-a_0 is small, and therefore, if we put $l_0=a_0+\epsilon$ and neglect ϵ^2, it can be shown that

$$m-m_0=0,$$
$$l-l_0=-\frac{4}{\gamma+1}\epsilon,$$
$$p=\frac{4}{\gamma+1}\rho_0 a_0\epsilon,$$
$$\rho=\frac{4}{\gamma+1}\frac{\rho_0}{a_0}\epsilon,$$

and consequently the shock condition becomes

$$m-m_0:l-l_0:p:\rho=0:-a_0:\rho_0 a_0^2:\rho_0$$

on S, the shock curve. The equation of S is $y=x\tan\mu_0+d(x,\ t)$, where μ_0 is the Mach angle for the initial uniform stream and d is small. It follows that, if

$\phi(x, y, t)$ is also small, its value on S is equal to $\phi(x, x\tan\mu_0, t)$, correct to the first order of small quantities. This means that the value of ϕ on S can be replaced by its value on L, the line $y = x\tan\mu_0$. Thus, the shock condition is

$$u : v : p : \rho = -a_0 \sin\mu_0 : a_0 \cos\mu_0 : \rho_0 a_0^2 : \rho_0 \text{ on } L \quad . \quad . \quad \text{(B)}$$

because $\psi - \mu_0$ is a first order term, ψ being the angle which the tangent to S at P makes with the x-axis. As $p = a_0^2\rho$ on L, it is seen from (A) that $f = 0$ on L. Therefore $f(x - u_0 t, x\tan\mu_0) = 0$, and consequently $f \equiv 0$, since $x - u_0 t$ and $x\tan\mu_0$ are independent. The formulae for $\mathbf{u}$, p and ρ may now be written

$$\left.\begin{aligned} &\mathbf{u} = \operatorname{grad} H + \operatorname{curl}(\alpha \mathbf{k}), \\ &p = a_0^2\rho = -\rho_0\left(\frac{\partial H}{\partial t} + u_0 \frac{\partial H}{\partial x}\right). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad \text{(C)}$$

The boundary condition at the aerofoil is that $v = \omega x + u_0\theta + v_0$ on OA, and the same reasoning as that applied to the shock line proves that OA may be approximated to $y = 0$. Thus

$$v = \omega x + u_0\theta + v_0 \text{ on } y = 0. \quad . \quad . \quad . \quad . \quad \text{(D)}$$

The problem is now specified by (B, (C), (D) and (4), and it will be proved that Temple and Jahn's solution satisfies all these conditions. Firstly, it satisfies (D), (C) and (4) because it is so constructed, and it therefore remains to be shown that it also satisfies (B). In the solution in question $\alpha \equiv 0$ and $H = 0$ on L.

Therefore $$\frac{\partial H}{\partial t} = 0 \text{ and } \frac{\partial H}{\partial x} + \frac{\partial H}{\partial y}\tan\mu_0 = 0 \text{ on } L.$$

Thus $$\frac{\partial H}{\partial x} : \frac{\partial H}{\partial y} : \frac{\partial H}{\partial t} = -\sin\mu_0 : \cos\mu_0 : 0 \text{ on } L \text{ and therefore}$$

$$u : v : p : \rho = -a_0 \sin\mu_0 : a_0 \cos\mu_0 : \rho_0 : \rho_0 a_0^2 \text{ on } L.$$

Hence (B) is satisfied, which means that Temple and Jahn's solution satisfies completely this linearised theory.

Fig. 2.

On applying the above theory to a curved aerofoil, (D) is the only condition which undergoes any change.

Let the upper surface be of the form $y=F(x)$ when the chord OA coincides with the x-axis. Thus $PN=F(ON)$, and since the aerofoil is slender, $F(x)$ is small. The inclination of the direction of the velocity at P to the x-axis is therefore $\theta+F'(x)$, so that the condition (D) is replaced by

$$v=\omega x+u_0\theta+v_0+u_0F'(x) \text{ on } y=0.$$

Consequently, in order to obtain the complete solution for a curved aerofoil, we merely superpose the solution for a steady aerofoil of the same form on that for a flat aerofoil of the same length oscillating in the same manner.

ACKNOWLEDGMENT

The author wishes to express his thanks to Professor G. Temple for his aid in producing this research, and also to the Department of Scientific and Industrial Research for financial assistance.

REFERENCES

1. TEMPLE, G., and JAHN, H. A. (1945). Flutter at Supersonic Speeds. R. & M. 2140, April 1945.

2. CARRIER, G. F. (1949). On the Stability of Supersonic Flow past a Wedge. *Quarterly of Applied Mathematics*, Vol. VI, No. 4, January 1949.

THE LINEARISED THEORY OF CONICAL FIELDS IN SUPERSONIC FLOW, WITH APPLICATIONS TO PLANE AEROFOILS

by

S. GOLDSTEIN, F.R.S., F.R.Ae.S.,
and
G. N. WARD
(*Department of Mathematics, The University, Manchester*)

SUMMARY

In many important problems of supersonic flow, either for the whole field of flow or a part of it, the velocity components are constant on straight lines through a fixed point. Such velocity fields are called conical fields. In the usual theory of the linearised perturbations of a steady supersonic flow, the velocity is assumed to differ only slightly from a uniform undisturbed velocity, and in the defining equations and boundary conditions all non-linear terms in the components of the perturbation velocity (and their space derivatives) are neglected. In this paper the equations of linearised supersonic conical fields, and their general solution, are set out both for the region inside and for the region outside the Mach cone of the origin in the conical field. The results are applied to flow past plane triangular aerofoils with straight edges downstream of the vertex (of which there are six cases), to flow past those plane aerofoils of more extended shape and finite span for which the solution may be obtained by combining a finite number of conical fields, and to the problem of plane triangular vanes in semi-infinite free jets. In the applications, the velocity fields are determined in considerable detail, but a main purpose in setting them out was to exhibit the mathematical methods used and the physical considerations that enter in determining the mathematical solutions.

Notation

a, b, c, d	in Sections 9, 10, 11, integers
f_1, f_2, f_3	arbitrary periodic functions of $\sigma+\theta$
g_1, g_2, g_3	arbitrary functions of ζ

Paper received February 1949.
[The Aeronautical Quarterly, Vol. II, May 1950]

k	constant in Section 5
k, k'	modulus and complementary modulus of elliptic functions in Section 11
k_1	modulus of elliptic functions, defined in equation (139)
l, m, n	direction cosines of normal to surface of body
m	in Section 11, defined by equation (148)
m	in Sections 7, 9, 10, an integer
n	in Section 7, an integer
p	pressure of undisturbed stream; in equations (37), (38), (44) and (45), variable of integration
Δp	perturbation pressure
r, θ	polar co-ordinates in (x_1, y_1) plane
s	real variable defined in equation (21)
t	complex variable defined in equation (60)
t_0, t_1, t_2	values of t, being the limits of x_1 on aerofoils. (Equations (64), (110))
u, v, w	perturbation velocity components
x, y, z	rectangular Cartesian co-ordinates. (Section 1)
x', y, z'	rectangular Cartesian co-ordinates after rotation of axes. (Section 3)
x_1, y_1	rectangular co-ordinates in the plane $z = \cot \mu$. (Equation (14))
A_1, A_2, A_3, A_4, C_4	real constants
$E(\chi), E$	incomplete and complete elliptic integrals of the second kind to modulus k
E_1	complete elliptic integral of the second kind to modulus k_1
$F(\chi), F$	incomplete and complete elliptic integrals of the third kind, defined in equations (150) and (152)
F_1, F_2, F_3	arbitrary periodic functions of $\sigma - \theta$
G_1, G_2, G_3	arbitrary functions of t
K	real constant in Section 6
K, K'	complete elliptic integrals of the first kind to moduli k, k' in Section 11

M	Mach number of undisturbed stream
S_a, S_b, etc.	symbolic conical-field solutions in Section 12
W	velocity of undisturbed stream
α	angle of incidence, defined in Section 1
β_0, β_1, β_2	angles of sweep of leading or trailing edges of aerofoils
ε	arbitrary (small, positive) number, tending to zero
ζ	complex variable defined in equation (23)
θ	polar angle in (x_1, y_1) plane
λ	variable of integration
μ	Mach angle
μ_0, μ_1	angles defined in equations (13) and (47)
ρ	density of undisturbed stream
σ	variable defined in equation (17)
σ_0, σ_1	values of σ at leading or trailing edges of aerofoils, defined in equation (40)
ϕ	perturbation velocity potential
χ	complex variable defined in equation (132)
ψ	variable of integration
$\mathscr{R}$	" the real part of "
$\mathscr{I}$	" the imaginary part of "

1. INTRODUCTION. *Linear perturbations of steady supersonic flow*

In the usual theory of the linearised perturbations of the steady supersonic flow of a gas, a velocity field is considered in which the velocity is supersonic everywhere. Viscosity, heat conduction, and heat radiation are neglected. The entropy of the gas per unit mass is assumed to be constant throughout the whole field. External forces, and therefore all variations in the potential energy per unit mass throughout the field, are neglected. The stagnation enthalpy (the sum of the enthalpy and kinetic energy) per unit mass, which is then constant along any streamline, is assumed to be constant throughout the field, and the flow is irrotational (apart from the possible appearance of vortex sheets and singularities, as discussed in Section 6 *et seq.*). The velocity is assumed to differ only slightly from a uniform undisturbed velocity, and in the defining equations and boundary conditions all

terms are neglected that are of the second or higher degree in the components of the perturbation velocity and their space derivatives.

Let (x, y, z) be rectangular Cartesian co-ordinates in the field, with the axis of z in the direction of the undisturbed velocity, whose magnitude is denoted by W. Let the undisturbed Mach number, equal to the ratio of W to the undisturbed acoustic velocity, be M, and let the Mach angle be μ, so that μ is the acute angle for which

$$M \sin \mu = 1, \quad \cot^2 \mu = M^2 - 1. \qquad (1)$$

Denote the velocity components by $u, v, W+w$, and let ϕ be the perturbation velocity potential, so that

$$u = \partial\phi/\partial x, \quad v = \partial\phi/\partial y, \quad w = \partial\phi/\partial z. \qquad (2)$$

Also denote by p and ρ the undisturbed pressure and density of the gas, and by $p+\Delta p$ the pressure at any point.

From the equation of continuity, u, v, w, and ϕ all satisfy the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \cot^2\mu \frac{\partial^2\phi}{\partial z^2} \qquad (3)$$

to a linear approximation and, from the integral of the equations of motion,

$$\Delta p = -\rho W w, \qquad (4)$$

also to a linear approximation.

We now assume that the disturbance in the flow is caused by the presence of a solid body, and, in specifying boundary conditions, we consider first the conditions at its surface. In some cases (*e.g.* for aerofoils) it is desirable to separate the effect of incidence, and therefore to begin with a precise definition of incidence. Generally the origin of co-ordinates will be taken as a point on the surface of the body such that no part of the surface is upstream of the origin, in $z < 0$. (The origin is so chosen in all subsequent sections of this paper except at the end of Section 10.) For definiteness, the axes of x and y are assumed chosen so that they form a right-handed system with the axis of z, but may otherwise be chosen as most convenient. An incidence α is defined as a right-handed rotation about the x-axis through an angle α from a specified standard position, and this definition is taken into account in choosing the axes of x and y; for example, if the surface of the solid body in its standard position is near a plane through the axis of z, this plane is taken to be $y=0$. In the standard position, let l, m, n be the direction cosines of the normal to

the surface drawn through a point of the surface into the fluid; then at incidence α the direction cosines are

$$l,\ m\cos\alpha - n\sin\alpha,\quad n\cos\alpha + m\sin\alpha.$$

To accord with a linear approximation α must be small, and when only first powers of α are retained the condition that the normal velocity is zero at the surface is

$$lu + (m - n\alpha)v + (n + m\alpha)(W + w) = 0. \quad . \quad . \quad . \quad (5)$$

Since u, v, w are small compared with W, the multiplier of W must be small. The term in w and the term $n\alpha v$ may be omitted, and the surface must be such that n is small. The boundary condition (5) becomes

$$lu + mv = -(n + m\alpha)W, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and, since α is small, it may be applied at the standard position of the surface, $\alpha = 0$.

When the surface is near the plane $y = 0$, and has small slopes (*e.g.* for thin aerofoils with flat or nearly flat boundaries), l and n are small, and $1 - m$ is small of the second order. The boundary condition (6) then simplifies to

$$v = -(n + \alpha)W, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and may be applied on the part of the plane $y = 0$ that is the projection of the surface.

Other boundary conditions are derived from the consideration that with no part of the bounding surface upstream of the origin there is no disturbance for $z < 0$, and u, v, w, and ϕ all vanish at $z = 0$ (except perhaps at the origin and at any other points in the bounding surface in the plane $z = 0$). Further, if no section of the bounding surface by a plane $z = constant$ reaches to infinity, the disturbance will have reached only a finite distance for any finite value of z, and, in any plane $z = constant$, u, v, w, ϕ, and all their derivatives, vanish beyond a certain finite distance from the axis of z.

2. INTRODUCTION. *Conical fields*

In many important problems of supersonic flow, either for the whole field of flow or a part of it, no fundamental length can be specified, and the velocity components are constant on straight lines through a fixed point. With this point as origin, u, v, w are functions of the angular variables only in a system of spherical polar co-ordinates; in Cartesian co-ordinates u, v, w are homogeneous functions of x, y, z of degree zero. Such velocity fields are called conical fields; their existence

was first recognised by Busemann[3]. When linear approximations are made, the mathematical analysis of such fields is particularly simple and elegant. There are other powerful and more general methods of considering solutions of the equations of Section 1; we are not here concerned with these other methods (to which some references are given at the end). The practical importance of conical fields, and the simplicity and elegance of their mathematics, entitle them to a separate detailed discussion.

In the following sections we set out the equations of linearised supersonic conical fields and their general solution, and we apply the results to flow along plane triangular aerofoils with straight edges lying downstream of the vertex (which is taken as the origin), to flow past some plane aerofoils of a more extended shape, and to flow past plane triangular vanes in semi-infinite free jets. At zero incidence the aerofoils are in the plane $y=0$, so $n=0$ on both the upper and lower surfaces, and the boundary condition in (7) is simply

$$v=-W\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

on the part of the plane $y=0$ occupied by the aerofoil at zero incidence; the boundary condition is the same for both the upper and lower surfaces.* In such cases v is an even function, and u and w are odd functions, of y.

Many results for plane aerofoils have been obtained by other authors, but the methods used here for obtaining both the fundamental relations for conical fields and the solutions for plane aerofoils are believed not to have been previously published in their entirety in the form here presented. The solutions for plane aerofoils seem to be more complete than those usually presented, and the results for plane vanes in a free jet are new. A list of references to the work of other authors is given at the end of this paper. Special mention should be made here of the work of W. D. Hayes, to which the attention of one of us (S.G.) was drawn when the main contents of this paper were presented in January 1948 in a lecture at the California Institute of Technology; Hayes's methods have much in common with ours. The overlap with the works of other authors is much less.

* The methods used in this paper may also be applied to more complicated examples, and a few such applications have been made by our co-workers. One such application is to the problem of the triangular aerofoil with finite dihedral and with edges swept back inside the Mach cone of the vertex. The aerofoil consists of two plane portions, which meet along the axis of z at an angle $\pi-2\delta$ and are each inclined, at zero incidence, at an angle δ to the plane $y=0$. The boundary condition (6) is

$$\mp u\sin\delta+v\cos\delta=-W\alpha\cos\delta \text{ on } x\sin\delta\mp y\cos\delta=0 \text{ for } x\gtrless 0$$

for both the upper and lower surfaces. The conditions apply on the parts of the planes $x\sin\delta\mp y\cos\delta=0$ occupied by the aerofoil at zero incidence, and δ has been taken as positive if the parts of the aerofoil are bent upwards (towards the axis of y) from the plane $y=0$.

3. FLOW OVER AN INFINITE PLANE YAWED WING WITH A STRAIGHT LEADING EDGE

It will be convenient to set out here for reference the results for an infinite plane wing with a straight leading edge.

When the leading edge is perpendicular to the undisturbed stream, it is taken as the axis of x; the solution is independent of x, and $u=0$. The solution,* which is well known, is, for $y>0$,

$$v=w=0 \text{ for } z<-y\cot\mu, \quad v=-W\alpha, \quad w=W\alpha\tan\mu \text{ for } z\geqslant y\cot\mu, \quad . \qquad (9)$$

and, for $y<0$,

$$v=w=0 \text{ for } z<y\cot\mu, \quad v=-W\alpha, \quad w=-W\alpha\tan\mu \text{ for } z\geqslant -y\cot\mu. \qquad (10)$$

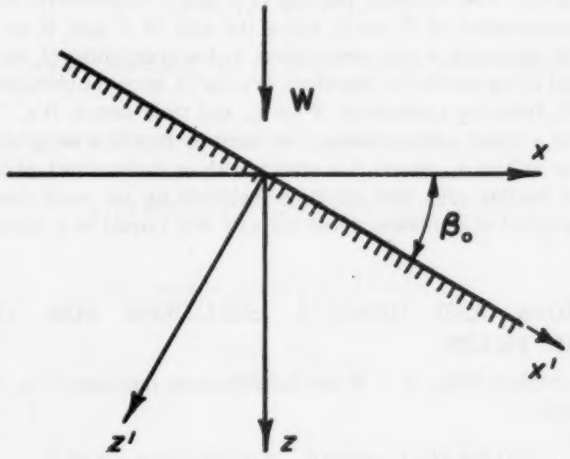

Fig. 1.

If the aerofoil is yawed through an angle β_o ($<\frac{1}{2}\pi-\mu$), so that its leading edge is along the axis of x' (Fig. 1), the solution* is, for $y>0$,

$$\left.\begin{aligned} &u=v=w=0 \text{ for } z\cos\beta_o < y\cot\mu_o + x\sin\beta_o, \\ &u=-W\alpha\sin\beta_o\tan\mu_o, \quad v=-W\alpha, \quad w=W\alpha\cos\beta_o\tan\mu_o \\ &\qquad\qquad \text{for } z\cos\beta_o \geqslant y\cot\mu_o + x\sin\beta_o, \end{aligned}\right\} \qquad (11)$$

* These solutions and the solutions in the following sections do not hold downstream of the trailing edges of the aerofoils.

and, for $y<0$,

$$\left.\begin{aligned} u=v=w=0 \quad & \text{for } z\cos\beta_0 < -y\cot\mu_0 + x\sin\beta_0, \\ u=W\alpha\sin\beta_0\tan\mu_0, \quad v=-W\alpha, \quad & w=-W\alpha\cos\beta_0\tan\mu_0 \\ & \text{for } z\cos\beta_0 \geqslant -y\cot\mu_0 + x\sin\beta_0. \end{aligned}\right\} \quad (12)$$

In this solution μ_0 is the acute angle defined by

$$M\cos\beta_0\sin\mu_0=1, \quad \cot^2\mu_0=M^2\cos^2\beta_0-1=\cot^2\mu\cos^2\beta_0-\sin^2\beta_0, \quad . \quad (13)$$

and the incidence α is, by definition, a positive rotation about the axis of x, not about the axis of x'. The solution may be found by finding the relevant solution for ϕ depending on y and z' only, and independent of x', with the boundary condition $v=-W\alpha$ on $y=0$ for $z'>0$. Alternatively, we may note that the incidence α involves rotations $\alpha\cos\beta_0$, $-\alpha\sin\beta_0$, about the axes of x' and z' respectively; the undisturbed velocity is compounded of $W\cos\beta_0$ along the axis of z' and $W\sin\beta_0$ along the axis of x'. At incidence α the contribution to the component of the undisturbed velocity normal to the aerofoil is, therefore, $W\alpha\cos^2\beta_0$ from the component $W\cos\beta_0$, and $W\alpha\sin^2\beta_0$ from the component $W\sin\beta_0$, and their sum is $W\alpha$. The problem is therefore, to a linear approximation, the same as that of a wing with its leading edge along the axis of x', placed in a stream with an undisturbed velocity $W\cos\beta_0$ normal to the leading edge, and set at an incidence (in the usual sense for a two-dimensional aerofoil of a rotation about the axis of x') equal to $\alpha/\cos\beta_0$.

4. EQUATIONS AND GENERAL SOLUTIONS FOR LINEARISED CONICAL FIELDS

Since, in conical fields, u, v, w are homogeneous functions of x, y, z of degree zero, if we write

$$x/(z\tan\mu)=x_1=r\cos\theta, \quad y/(z\tan\mu)=y_1=r\sin\theta, \quad . \quad . \quad (14)$$

then u, v, w are functions of x_1 and y_1 only, and of r and θ only.

In terms of r and θ the equations (of the form (3)) satisfied by u, v, and w become

$$\left[r^2(r^2-1)\frac{\partial^2}{\partial r^2}+r(2r^2-1)\frac{\partial}{\partial r}\right](u,v,w)=\frac{\partial^2}{\partial\theta^2}(u,v,w). \quad . \quad . \quad (15)$$

For $r>1$ we write the equation (15) for u as

$$r(r^2-1)^{\frac{1}{2}}\frac{\partial}{\partial r}\left[r(r^2-1)^{\frac{1}{2}}\frac{\partial u}{\partial r}\right]=\frac{\partial^2 u}{\partial\theta^2}, \quad . \quad . \quad . \quad (16)$$

and put $$1/r=\cos\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

so that $$\frac{\partial^2 u}{\partial\sigma^2}=\frac{\partial^2 u}{\partial\theta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

As r increases from 1 to ∞, σ increases from 0 to $\frac{1}{2}\pi$.

The general solution for u in $r>1$ is

$$u=f_1(\sigma+\theta)+F_1(\sigma-\theta), \quad . \quad . \quad . \quad . \quad (19a)$$

where f_1, F_1 are arbitrary functions. Similarly

$$v=f_2(\sigma+\theta)+F_2(\sigma-\theta), \quad . \quad . \quad . \quad . \quad (19b)$$

$$w=f_3(\sigma+\theta)+F_3(\sigma-\theta). \quad . \quad . \quad . \quad . \quad (19c)$$

Since u, v, w are single-valued, the f's and F's must be periodic functions with period 2π.

For $r<1$ we write the equation (15) for u as

$$r(1-r^2)^{\frac{1}{2}}\frac{\partial}{\partial r}\left[r(1-r^2)^{\frac{1}{2}}\frac{\partial u}{\partial r}\right]+\frac{\partial^2 u}{\partial\theta^2}=0, \quad . \quad . \quad . \quad (20)$$

and put $$s=-\cosh^{-1}(1/r)=\log\frac{1-(1-r^2)^{\frac{1}{2}}}{r}, \quad . \quad . \quad . \quad (21)$$

so that $$\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial\theta^2}=0. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Now let

$$\zeta=e^{s+i\theta}=\frac{1-(1-r^2)^{\frac{1}{2}}}{r}e^{i\theta}=\frac{r}{1+(1-r^2)^{\frac{1}{2}}}e^{i\theta}. \quad . \quad . \quad . \quad (23)$$

Then for $r<1$,

$$u=\mathscr{R}\,g_1(\zeta), \quad . \quad . \quad . \quad . \quad . \quad . \quad (24a)$$

where $\mathscr{R}$ denotes "the real part of," and $g_1(\zeta)$ is an arbitrary function. Similarly

$$v=\mathscr{R}\,g_2(\zeta), \quad w=\mathscr{R}\,g_3(\zeta). \quad . \quad . \quad . \quad . \quad (24b, c)$$

When $r=1$, $s=0$ and $e^s=1$; as r decreases from 1 to 0, $1/r$ increases from 1 to ∞, s decreases from 0 to $-\infty$, and e^s decreases from 1 to 0. Hence the interior of the

unit circle $r=1$ in the (r, θ) plane corresponds with the interior of the unit circle $|\zeta|=1$ in the ζ-plane; the boundaries correspond, and θ is the same in both planes.

We have still to take account of the conditions that u, v, w shall be the components of the gradient of a function ϕ, which itself satisfies equation (3). Since u, v, w are homogeneous functions of degree zero in x, y, and z,

$$xu+yv+zw=x\partial\phi/\partial x+y\partial\phi/\partial y+z\partial\phi/\partial z=\phi, \quad . \quad . \quad . \quad (25)$$

so

$$d(xu+yv+zw-\phi)=xdu+ydv+zdw=0. \quad . \quad . \quad . \quad (26)$$

If $g_1(\zeta)=u+iu_1$, the component of the gradient of u_1 in any direction is equal to the component of the gradient of u in a direction at right angles, and similarly for v and w. Hence equation (26) implies that

$$xdu_1+ydv_1+zdw_1=0,$$

and

$$xg_1'(\zeta)+yg_2'(\zeta)+zg_3'(\zeta)=0. \quad . \quad . \quad . \quad . \quad (27)$$

Differentiation of (27) with respect to x, y, and z in turn leads to the equation

$$[xg_1''(\zeta)+yg_2''(\zeta)+zg_3''(\zeta)]\partial\zeta/\partial x+g_1'(\zeta)=0, \quad . \quad . \quad (28)$$

and two similar equations. It follows that

$$g_1'(\zeta):g_2'(\zeta):g_3'(\zeta)=\frac{\partial\zeta}{\partial x}:\frac{\partial\zeta}{\partial y}:\frac{\partial\zeta}{\partial z}=\frac{\partial}{\partial x}(s+i\theta):\frac{\partial}{\partial y}(s+i\theta):\frac{\partial}{\partial z}(s+i\theta). \quad (29)$$

By expressing derivatives with respect to x, y, and z in terms of derivatives with respect to r and θ, it may be proved that

$$\left.\begin{aligned} 2z\tan\mu\frac{\partial}{\partial x}(s+i\theta)&=(1-r^2)^{-\frac{1}{2}}\left(\zeta+\frac{1}{\zeta}\right),\\ 2z\tan\mu\frac{\partial}{\partial y}(s+i\theta)&=-i(1-r^2)^{-\frac{1}{2}}\left(\zeta-\frac{1}{\zeta}\right),\\ 2z\frac{\partial}{\partial z}(s+i\theta)&=-2(1-r^2)^{-\frac{1}{2}}. \end{aligned}\right\} \quad . \quad . \quad . \quad (30)$$

Hence

$$g_1'(\zeta):g_2'(\zeta):g_3'(\zeta)=-(\zeta^2+1):i(\zeta^2-1):2\zeta\tan\mu, \quad . \quad . \quad (31)$$

and the general solution for u, v, and w in $r<1$ is

$$\left.\begin{aligned} u &= -\mathscr{R}\int^{\zeta}(\lambda^2+1)\,g(\lambda)\,d\lambda, \\ v &= \mathscr{R}\,i\int^{\zeta}(\lambda^2-1)\,g(\lambda)\,d\lambda, \\ w &= 2\tan\mu\,\mathscr{R}\int^{\zeta}\lambda g(\lambda)\,d\lambda, \end{aligned}\right\} \quad . \quad . \quad . \quad (32)$$

where $g(\lambda)$ is an arbitrary function of λ.

Since ζ is a homogeneous function of degree zero in x, y, and z,

$$x\partial\zeta/\partial x+y\partial\zeta/\partial y+z\partial\zeta/\partial z=0, \quad . \quad . \quad . \quad . \quad (33)$$

so

$$xg_1'(\zeta)+yg_2'(\zeta)+zg_3'(\zeta)=0, \quad . \quad . \quad . \quad . \quad (34)$$

from which it is easily checked that, if $\phi=xu+yv+zw$,
then $(u, v, w)=(\partial/\partial x, \partial/\partial y, \partial/\partial z)\phi$, and also (from (31)) that ϕ itself satisfies the equation (3).

When $r>1$, by use of the same arguments the relations

$$\left.\begin{aligned} xf_1'(\sigma+\theta)+yf_2'(\sigma+\theta)+zf_3'(\sigma+\theta)&=0, \\ xF_1'(\sigma-\theta)+yF_2'(\sigma-\theta)+zF_3'(\sigma-\theta)&=0, \end{aligned}\right\} \quad . \quad . \quad . \quad (35)$$

are found for the derivatives of the functions in equation (19), and from these relations it is deduced that

$$\left.\begin{aligned} f_1' : f_2' : f_3' &= \frac{\partial}{\partial x}(\sigma+\theta) : \frac{\partial}{\partial y}(\sigma+\theta) : \frac{\partial}{\partial z}(\sigma+\theta) \\ &= \cos(\sigma+\theta) : \sin(\sigma+\theta) : -\tan\mu, \\ F_1' : F_2' : F_3' &= \frac{\partial}{\partial x}(\sigma-\theta) : \frac{\partial}{\partial y}(\sigma-\theta) : \frac{\partial}{\partial z}(\sigma-\theta) \\ &= \cos(\sigma-\theta) : -\sin(\sigma-\theta) : -\tan\mu. \end{aligned}\right\} \quad . \quad (36)$$

Hence the general solution for u, v, and w in $r>1$ is

$$\left.\begin{aligned} u &= \int^{\sigma+\theta} f(p)\cos p\,dp + \int^{\sigma-\theta} F(p)\cos p\,dp, \\ v &= \int^{\sigma+\theta} f(p)\sin p\,dp - \int^{\sigma-\theta} F(p)\sin p\,dp, \\ w &= -\tan\mu\int^{\sigma+\theta} f(p)\,dp - \tan\mu\int^{\sigma-\theta} F(p)\,dp, \end{aligned}\right\} \quad . \quad . \quad . \quad (37)$$

where $f(p)$ and $F(p)$ are periodic functions of period 2π such that

$$\int_0^{2\pi} f(p)\,dp = \int_0^{2\pi} f(p)\cos p\,dp = \int_0^{2\pi} f(p)\sin p\,dp = 0, \quad . \quad . \quad (38)$$

and similarly for $F(p)$.

5. THE VELOCITY COMPONENTS OUTSIDE THE MACH CONE OF THE ORIGIN*

In an application to the flow over a plane triangular aerofoil the boundary condition $v=-W\alpha$ is to be applied on the part of the plane $y=0$ occupied by the aerofoil at zero incidence, which corresponds to a certain portion of the line $y_1=0$ in the (x_1, y_1) plane. This portion may or may not lie, wholly or partly, outside the circle $r=1$; in this section we consider the cases when it does.

The part of the (x_1, y_1) plane for which $r\geqslant 1$ corresponds in the (σ, θ) plane to a strip between $\sigma=0$ and $\sigma=\frac{1}{2}\pi$. The axis $y_1=0$ corresponds to $\theta=0$ (for $x_1>0$) and to $\theta=\pi$ (for $x_1<0$), so the boundaries in the (σ, θ) plane corresponding to those parts of a plane triangular aerofoil that are outside the Mach cone of its vertex lie along $\theta=0$ and $\theta=\pi$ (and are repeated at intervals of 2π). They extend from $\sigma=0$ ($r=1$) if the aerofoil passes through the Mach cone, but certainly do not reach to $\sigma=\frac{1}{2}\pi$ ($r=\infty$). We consider first the case in which the boundaries extend from $\sigma=0$ to $\sigma=\sigma_0$ on $\theta=0$ and to $\sigma=\sigma_1$ on $\theta=\pi$ ($\sigma_0<\frac{1}{2}\pi$, $\sigma_1<\frac{1}{2}\pi$, *see* Fig. 2*a*). Since

$$(x/z)\cot\mu = x_1 = r\cos\theta = \sec\sigma\cos\theta, \quad . \quad . \quad . \quad (39)$$

this case is that of a triangular aerofoil whose edges are outside and on opposite sides of the Mach cone and are swept back at angles β_0, β_1, with $\beta_0<\frac{1}{2}\pi-\mu$, $\beta_1<\frac{1}{2}\pi-\mu$ (Fig. 2*b*), and

$$\sec\sigma_0=\cot\beta_0\cot\mu, \quad \sec\sigma_1=\cot\beta_1\cot\mu. \quad . \quad . \quad . \quad (40)$$

* The part downstream of any point P of the right circular cone with vertex at P and axis in the direction of the z-axis, and with semi-vertical angle equal to the Mach angle μ will be called the Mach cone of P. The Mach cone of the origin corresponds with the circle $r=1$ in the (x_1, y_1) plane.

Now near $\sigma=\frac{1}{2}\pi$,

$$v=\partial v/\partial\sigma=\partial v/\partial\theta=0, \quad . \quad . \quad . \quad . \quad . \quad (41)$$

since there is no disturbance beyond some finite value of r. Hence, from (19*b*),

$$f_2(\sigma+\theta)+F_2(\sigma-\theta)=f_2'(\sigma+\theta)+F_2'(\sigma-\theta)=f_2'(\sigma+\theta)-F_2'(\sigma-\theta)=0, \quad (42)$$

so

$$f_2'(\sigma+\theta)=F_2'(\sigma-\theta)=0$$

and

$$f_2(\sigma+\theta)=-F_2(\sigma-\theta)=C \quad . \quad . \quad . \quad . \quad (43)$$

near $\sigma=\frac{1}{2}\pi$, where C is a constant. Now $f_2(\sigma+\theta)$ and $F_2(\sigma-\theta)$ are constant along the lines $\sigma+\theta=$*constant* and $\sigma-\theta=$*constant*, respectively. (These lines are the characteristics of the differential equation (18).) If lines $\sigma+\theta=$*constant* and $\sigma-\theta=$*constant* can be drawn from any point P to $\sigma=\frac{1}{2}\pi$ without intersecting the cuts from $\sigma=0$ to $\sigma=\sigma_0$ along $\theta=0$ and from $\sigma=0$ to $\sigma=\sigma_1$ along $\theta=\pi$, then equations (43) still hold, and $v=0$, at P. Thus v, and similarly u and w, vanish everywhere for $r\geqslant 1$ except in the regions labelled 1, 2, 3, 4, in Fig. 2*a*.

At the cuts the f's and F's may be discontinuous. Regions 1 and 3 may be reached from $\sigma=\frac{1}{2}\pi$ along lines $\sigma-\theta=$*constant* without a cut being crossed, so $F_2(\sigma-\theta)$ does not change as these regions are entered; but the regions cannot be entered from $\sigma=\frac{1}{2}\pi$ along lines $\sigma+\theta=$*constant* if a cut is not crossed, so $f_2(\sigma+\theta)$ may change discontinuously. In order that v should be equal to $-W\alpha$ on the cuts, $f_2(\sigma+\theta)$ must, in fact, decrease discontinuously by $W\alpha$ as $\sigma+\theta$ decreases through σ_0 when region 3 is entered, and as $\sigma+\theta$ decreases through $\sigma_1+\pi$ when region 1 is entered. Similarly, as regions 2 and 4 are entered, $f_2(\sigma+\theta)$ does not change, but $F_2(\sigma-\theta)$ must decrease discontinuously by $W\alpha$ as $\sigma-\theta$ decreases through σ_0, and through $\sigma_1-\pi$, respectively. Hence $v=W\alpha$ throughout all four regions. To find u and w, we use (37). For example, on entering region 3, $\int^{\sigma+\theta} f(p)\sin p\,dp$ decreases discontinuously by $W\alpha$ as $\sigma+\theta$ decreases through σ_0, so

$$\sin\sigma_0 \lim_{\epsilon\to 0}\int_{\sigma_0-\epsilon}^{\sigma_0+\epsilon} f(p)\,dp=W\alpha. \quad . \quad . \quad . \quad . \quad (44)$$

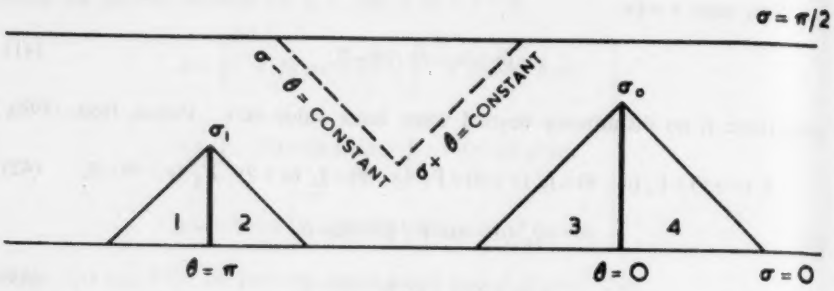

Fig. 2*a*.

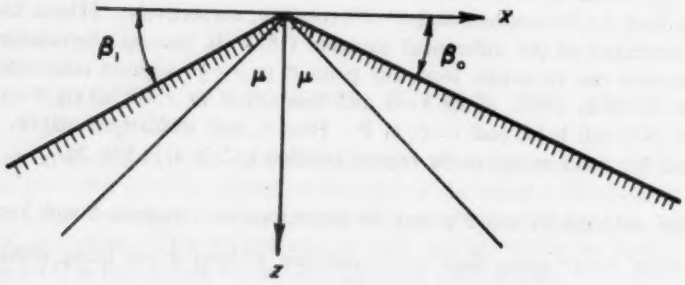

Fig. 2*b*.

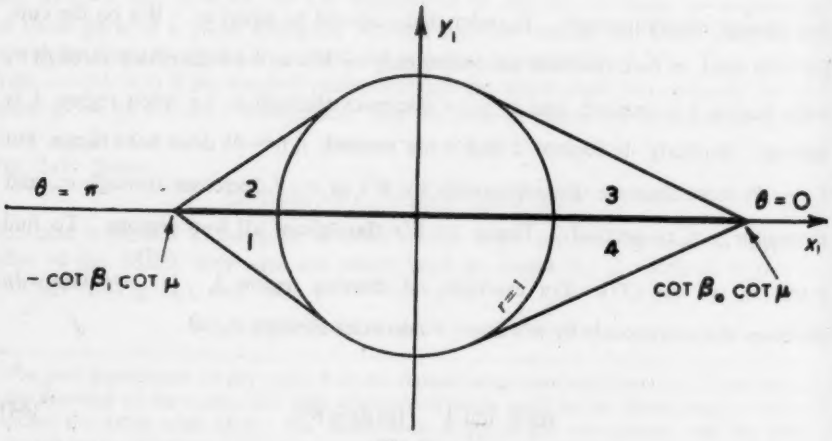

Fig. 2*c*.

The discontinuous decrease in u is

$$\cos\sigma_0 \lim_{\epsilon\to 0}\int_{\sigma_0-\epsilon}^{\sigma_0+\epsilon} f(p)\,dp = W\alpha\cot\sigma_0, \quad . \quad . \quad . \quad (45)$$

and that in w is similarly $-W\alpha\tan\mu\,\mathrm{cosec}\,\sigma_0$. Hence $u=-W\alpha\cot\sigma_0$ and $w=W\alpha\tan\mu\,\mathrm{cosec}\,\sigma_0$ throughout region 3. The values in the other regions are found similarly. The results are as follows:—

Region	u	v	w
1	$-W\alpha\cot\sigma_1$	$-W\alpha$	$-W\alpha\tan\mu\,\mathrm{cosec}\,\sigma_1$
2	$W\alpha\cot\sigma_1$	$-W\alpha$	$W\alpha\tan\mu\,\mathrm{cosec}\,\sigma_1$
3	$-W\alpha\cot\sigma_0$	$-W\alpha$	$W\alpha\tan\mu\,\mathrm{cosec}\,\sigma_0$
4	$W\alpha\cot\sigma_0$	$-W\alpha$	$-W\alpha\tan\mu\,\mathrm{cosec}\,\sigma_0$

. (46)

In these results v is an even function, and u and w are odd functions, of y, as they should be.

Now consider the corresponding regions in the (x_1, y_1) plane and in physical space. Since $r\cos\sigma=1$, $\sigma+\theta=k$ corresponds to the line $r\cos(k-\theta)=1$, which is the tangent to the circle $r=1$ at $\theta=k$. Similarly $\sigma-\theta=-k$ corresponds to the same tangent. Along any such tangent r decreases from ∞ to 1 and then increases again to ∞, and the complete tangent therefore corresponds to two characteristics in the (σ, θ) plane lying between $\sigma=0$ and $\sigma=\frac{1}{2}\pi$. Along one part of the tangent θ increases from $r=\infty$ to $r=1$; along the other part θ decreases from $r=\infty$ to $r=1$. These correspond to the characteristics in the (σ, θ) plane along which θ increases and decreases, respectively, as σ decreases from $\frac{1}{2}\pi$ to 0, and therefore to the characteristics $\sigma+\theta=k$, $\sigma-\theta=-k$, respectively. Thus the characteristics $\sigma+\theta=$*constant* correspond to those parts of the tangents to $r=1$ on which θ increases as the circle is approached, and the characteristics $\sigma-\theta=$*constant* to those parts of the tangents on which θ decreases as the circle is approached.

Now at zero incidence the aerofoil in Fig. 2*b* corresponds, in the (x_1, y_1) plane, to the part of the axis $y_1=0$ between $x_1=-\sec\sigma_1=-\cot\beta_1\cot\mu$ and $x_1=\sec\sigma_0=\cot\beta_0\cot\mu$, and the foregoing discussion shows that the regions 1, 2, 3, 4 in the (σ, θ) plane (Fig. 2*a*) correspond to the regions 1, 2, 3, 4 in the (x_1, y_1) plane (Fig. 2*c*), bounded by arcs of the circle $r=1$ (corresponding to $\sigma=0$), by $y_1=0$ (corresponding to $\theta=0$ and $\theta=\pi$), and by the tangents from the ends of the portion of $y_1=0$ corresponding to the aerofoil (*i.e.* from $x_1=-\sec\sigma_1$, $y_1=0$ and $x_1=\sec\sigma_0$, $y_1=0$, respectively).

Now let μ_0, μ_1 be the acute angles for which

$$\left.\begin{aligned} &M\cos\beta_0\sin\mu_0=1,\ \cot^2\mu_0=M^2\cos^2\beta_0-1=\cot^2\mu\cos^2\beta_0-\sin^2\beta_0,\\ \text{and}\quad &M\cos\beta_1\sin\mu_1=1,\ \cot^2\mu_1=M^2\cos^2\beta_1-1=\cot^2\mu\cos^2\beta_1-\sin^2\beta_1. \end{aligned}\right\} \quad (47)$$

Then
$$\left.\begin{aligned} \cot\sigma_0&=\sin\beta_0\tan\mu_0,\quad \operatorname{cosec}\sigma_0=\cot\mu\cos\beta_0\tan\mu_0,\\ \cot\sigma_1&=\sin\beta_1\tan\mu_1,\quad \operatorname{cosec}\sigma_1=\cot\mu\cos\beta_1\tan\mu_1, \end{aligned}\right\} \quad (48)$$

and the values of u, v, w in (46) may be written in terms of β_0, β_1, μ_0, μ_1. For regions 2 and 3 the values are as follows:—

Region	u	v	w
2	$W\alpha\sin\beta_1\tan\mu_1$	$-W\alpha$	$W\alpha\cos\beta_1\tan\mu_1$
3	$-W\alpha\sin\beta_0\tan\mu_0$	$-W\alpha$	$W\alpha\cos\beta_0\tan\mu_0$

(49)

In regions 1 and 4 the values of u and w are minus the values in regions 2 and 3, respectively, and $v=-W\alpha$. Outside the regions 1, 2, 3, 4, u, v, and w all vanish for $r\geqslant 1$.

These are the same formulæ as for infinite plane wings yawed through angles β_0, $-\beta_1$, respectively (Section 3): outside the Mach cone of the vertex of the triangular wing, the effect of the vertex is not felt, as we should expect. Moreover, it is correct that the tangent boundaries to the regions 1, 2, 3, 4 correspond to the plane "shock waves" or characteristic surfaces from the leading edges in physical space, whose equations (to a linear approximation) are (*cf.* equations (11) and (12))

$$z\cos\beta_0=x\sin\beta_0\pm y\cot\mu_0,\quad z\cos\beta_1=-x\sin\beta_1\pm y\cot\mu_1. \quad (50)$$

For example, the tangent boundary to region 3 is the line

$$x_1\cos\sigma_0+y_1\sin\sigma_0=1,\ \textit{i.e.}\ x_1\sin\beta_0+y_1\cot\mu_0=\cot\mu\cos\beta_0, \quad (51)$$

which corresponds to the plane

$$x\sin\beta_0+y\cot\mu_0=z\cos\beta_0$$

in physical space, and similarly for the other boundaries.

The relation between the "shock waves" from the leading edges in physical space, and the tangent boundaries in the (x_1, y_1) plane, may be seen in more detail

by another method. In the first place the "shock waves" are the envelopes of the Mach cones of points on the leading edge at zero incidence. The Mach cone of the point $(\lambda \cot \beta_0, 0, \lambda)$ on the leading edge in $x > 0$, for example, is

$$(x - \lambda \cot \beta_0)^2 + y^2 = (z - \lambda)^2 \tan^2 \mu, \quad . \quad . \quad . \quad . \quad (52)$$

and the envelope of this family of cones is the pair of planes

$$z \cos \beta_0 = x \sin \beta_0 \pm y \cot \mu_0. \quad . \quad . \quad . \quad . \quad (53)$$

Now (x_1, y_1) are the values of (x, y) on the plane $z = \cot \mu$, and any one of the above cones for which $0 \leqslant \lambda \leqslant \cot \mu$ meets this plane in the circle

$$(x_1 - \lambda \cot \beta_0)^2 + y_1^2 = (1 - \lambda \tan \mu)^2. \quad . \quad . \quad . \quad . \quad (54)$$

For $\lambda = 0$ (the vertex of the triangular aerofoil) this is the unit circle, $x_1^2 + y_1^2 = 1$; for $\lambda = \cot \mu$ it is the point $(\cot \mu \cot \beta_0, 0)$. For any intermediate value of λ it is the circle with centre at $(\lambda \cot \beta_0, 0)$ and of radius $1 - \lambda \tan \mu$; its centre is on the line joining the origin to $(\cot \mu \cot \beta_0, 0)$, and its radius is such that it touches the tangents to the unit circle from that point. The tangents therefore correspond to the envelopes of the cones in physical space. Moreover the insides of the cones correspond to the regions 1, 2, 3, 4 together with the inside of the unit circle—*i.e.* to the region in which u, v, w are not zero. (The region bounded by the continuations of the tangents from their points of contact to infinity, plus the unit circle, corresponds to the insides of the cones whose vertices are on the continuation of the leading edge into $z < 0$; *cf.* the end of Section 10.)

If one edge of the aerofoil is outside the Mach cone of the vertex and the other is inside, either regions 1 and 2 or regions 3 and 4 are absent, and the results for the remaining pair are unaltered, with $u = v = w = 0$ elsewhere for $r \geqslant 1$. If both edges are inside the Mach cone, $u = v = w = 0$ everywhere in $r \geqslant 1$.

6. THE VELOCITY COMPONENTS, OUTSIDE THE MACH CONE OF THE VERTEX, FOR A PLANE TRIANGULAR AEROFOIL ENTIRELY OUTSIDE THE MACH CONE

If the whole of a plane triangular aerofoil with swept-back edges is outside the Mach cone of its vertex, then both edges are outside the Mach cone, but on the same side of it (Fig. 3*a*; $\beta_0 < \beta_1 < \frac{1}{2}\pi - \mu$).

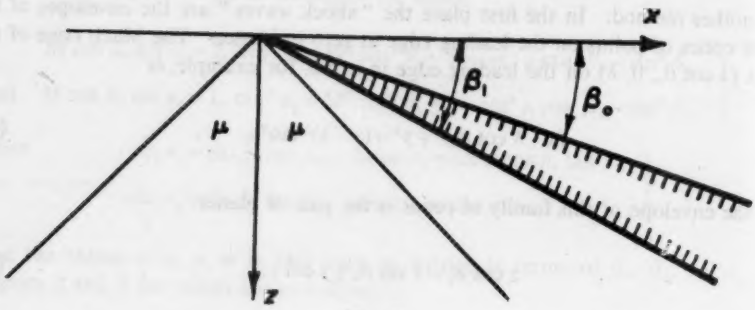

Fig. 3a.

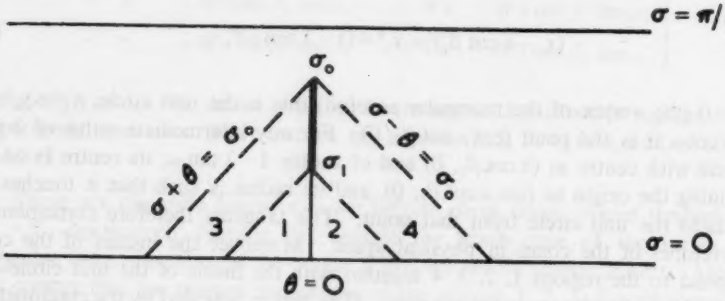

Fig. 3b.

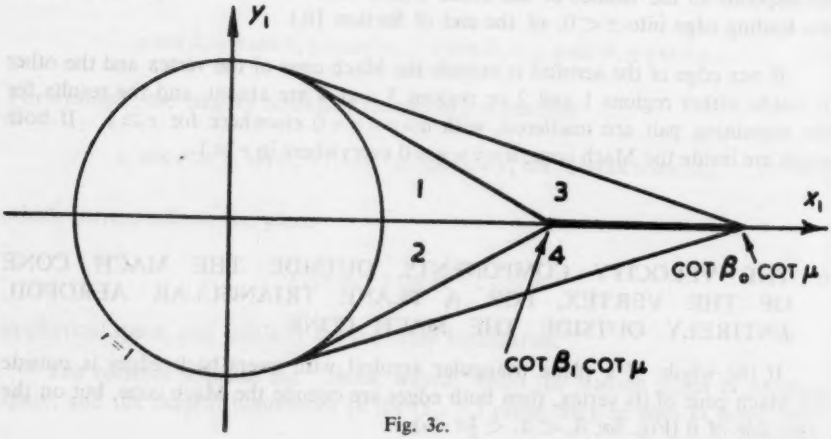

Fig. 3c.

The aerofoil surface at zero incidence corresponds, in the (σ, θ) plane, to the portion of $\theta=0$ for which $\sigma_1 \leqq \sigma \leqq \sigma_0$, with the same definitions of σ_0 and σ_1 as before (equation (40)). Except in the regions 1, 2, 3, 4 in Fig. 3*b*, in which the dashed lines are parts of $\sigma \pm \theta = \sigma_0$, $\sigma \pm \theta = \sigma_1$, u, v, and w are zero as before. The regions 1 and 2 are separated by $\theta=0$, $0 < \sigma < \sigma_1$, and on the boundary the normal velocity and the pressure must be continuous, *i.e.* v and w must be continuous.

As before, when the region 3 is entered, $f_2(\sigma+\theta)$ decreases discontinuously by $W\alpha$ as $\sigma+\theta$ decreases through σ_0, and $F_2(\sigma-\theta)$ does not change. When the region 4 is entered, $F_2(\sigma-\theta)$ decreases discontinuously by $W\alpha$ as $\sigma-\theta$ decreases through σ_0, and $f_2(\sigma+\theta)$ does not change. Thus v is constant and equal to $-W\alpha$ throughout regions 3 and 4. u and w are found as before, and their values are the same as in regions 3 and 4 in (46).

In passing into region 1 from region 3, $F_2(\sigma-\theta)$ does not change; let $f_2(\sigma+\theta)$ increase by K as $\sigma+\theta$ decreases through σ_1, so v becomes $-W\alpha+K$ throughout region 1. By the argument as before, u will increase by $K\cot\sigma_1$ and w by $-K\tan\mu\operatorname{cosec}\sigma_1$. Hence in region 1

$$u=-W\alpha\cot\sigma_0+K\cot\sigma_1,\quad v=-W\alpha+K,\quad w=\tan\mu\,(W\alpha\operatorname{cosec}\sigma_0-K\operatorname{cosec}\sigma_1). \tag{55}$$

In passing into region 2 from region 4, $f_2(\sigma+\theta)$ does not change, and $F_2(\sigma-\theta)$ must increase by K in order that v should be continuous on the boundary between regions 1 and 2. In region 2, therefore,

$$u=W\alpha\cot\sigma_0-K\cot\sigma_1,\quad v=-W\alpha+K,\quad w=-\tan\mu\,(W\alpha\operatorname{cosec}\sigma_0-K\operatorname{cosec}\sigma_1). \tag{56}$$

Since w must be continuous on the boundary between regions 1 and 2, $w=0$ in both these regions and

$$K=W\alpha\sin\sigma_1\operatorname{cosec}\sigma_0. \tag{57}$$

With the same definitions of μ_0 and μ_1 as before (equations (47)), equations (48) still hold, and in terms of β_0, β_1, μ_0, μ_1 the values of u, v, and w in region 1 are as follows:—

$$u=W\alpha\frac{\sin(\beta_1-\beta_0)\tan\mu_0}{\cos\beta_1},\quad v=-W\alpha\left(1-\frac{\tan\mu_0\cos\beta_0}{\tan\mu_1\cos\beta_1}\right),\quad w=0. \tag{58}$$

The values in region 3 are exactly as in (49). In regions 2 and 4, the values of u and w are minus the values in regions 1 and 3, respectively ($w=0$ in region 2), and the values of v are the same as those in 1 and 3, respectively.

The regions 1, 2, 3, 4 in the (x_1, y_1) plane are shown in Fig. 3*c*.

There is a discontinuity in u across the boundary between region 1 and region 2, so there is, downstream from the trailing edge, a vortex sheet of uniform strength (which is continued inside the Mach cone; *see* Section 9).

7. THE VELOCITY COMPONENTS INSIDE THE MACH CONE OF THE VERTEX

For $r \leqslant 1$, u, v, and w are the real parts of functions $g_1(\zeta)$, $g_2(\zeta)$, and $g_3(\zeta)$ of ζ (equation (24)). From the results of Sections 5 and 6, u, v, and w are known on the circle $r=1$ in the (x_1, y_1) plane, and therefore on the circle $|\zeta|=1$ in the ζ-plane. The part of the aerofoil, at zero incidence, inside the Mach cone of the vertex corresponds to the whole or part of the axis $y_1=0$ between $x_1=\pm 1$, and therefore to the whole or part of the real axis in the ζ-plane inside the circle $|\zeta|=1$; along this part of the real axis in the ζ-plane, $v=-W\alpha$. Hence, for $|\zeta| \leqslant 1$, the real part of $g_2(\zeta)$ is known on the unit circle $|\zeta|=1$ and on any cut there may be inside it, and we may find $g_2(\zeta)$ at all points inside the unit circle in the ζ-plane. If $g_2(\zeta)$ has no singularities for $|\zeta|<1$ and ζ not on the cut,* the solution for $g_2(\zeta)$ is unique. There are, however, cases in which the solution that is free from singularities inside the unit circle off the cut is not physically acceptable, and we must return to the question of the occurrence of such singularities; we may say at once that their occurrence is connected with the appearance of trailing vortex sheet, (Sections 9, 10, 11.) When $g_2(\zeta)$ has been determined, v is known; the relations (31), together with the values of u and w at any one point on the unit circle, enable u and w to be determined (and their values at other points on the unit circle may then be checked).

To solve for $g_2(\zeta)$ it is advantageous to transform from the ζ-plane to a t-plane by the conformal transformation

$$1/t=\tfrac{1}{2}(\zeta+1/\zeta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

From (23)

$$\frac{1}{t}=\frac{\cos\theta - i(1-r^2)^{\frac{1}{2}}\sin\theta}{r} \quad . \quad . \quad . \quad . \quad . \quad (60)$$

The circle $r=1$ and its interior correspond with the circle $|\zeta|=1$ and its interior; the circles are transformed, in the t-plane, into the upper and lower sides of the parts of the real axis for which $|t| \geqslant 1$ (Fig. 4); their interiors are represented on the whole of the t-plane with cuts along the portions $|t| \geqslant 1$ of the real axis, the upper halves $(0<\theta<\pi)$ being represented on the upper half-plane ($\mathscr{I}(t)>0$)

* The singularities on the cut are at its ends and satisfy the conditions listed, in particular condition (*ii*).

and the lower halves $(-\pi < \theta < 0)$ on the lower half-plane $(\mathscr{I}(t) < 0)$. ($\mathscr{I}$ denotes "the imaginary part of.") The axis $x_1 = 0$ and the imaginary axis in the ζ-plane, inside the unit circle in each case, correspond with one another, and are transformed into the imaginary axis in the t-plane.

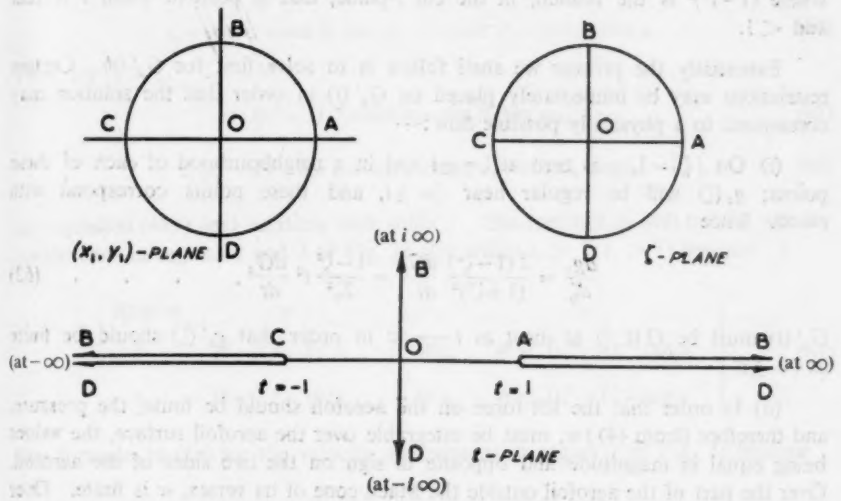

Fig. 4.

The point $(x_1, 0)$ on the x_1-axis inside the unit circle in the (x_1, y_1) plane corresponds (from (60)) with the point $t = x_1$ on the real axis in the t-plane. For a point on the unit circle in the (x_1, y_1) or ζ-plane, $t = \sec\theta$; at $\theta = \pm\sigma$, $t = \sec\sigma \pm 0i$, and $\sec\sigma$ is the abscissa, in the (x_1, y_1) plane, of the intersection with the x_1-axis of the tangent to the unit circle at $\theta = \pm\sigma$. Thus the point $t = x_1$ on the real axis in the t-plane corresponds with the point $(x_1, 0)$ if $|x_1| \leqslant 1$, and with the points of contact of the tangents to the unit circle from $(x_1, 0)$ if $|x_1| > 1$.

Now $v = -W\alpha$ on the part of the x_1-axis on which the aerofoil at zero incidence is represented, *i.e.* on this part of the x_1-axis, $v = -W\alpha$ at $(x_1, 0)$; and, if $|x_1| > 1$, it follows from Section 5 that $v = -W\alpha$ also at the points of contact of the tangents from $(x_1, 0)$ to the unit circle. Hence $v = -W\alpha$ on the real axis in the t-plane for all points with abscissæ between the extreme values of x_1 (*i.e.* of $x/(z\tan\mu)$) on the surface of the aerofoil at zero incidence.

Since u, v, w are the real parts of functions of ζ, they are also the real parts of functions of t. We put

$$u = \mathscr{R}G_1(t), \quad v = \mathscr{R}G_2(t), \quad w = \mathscr{R}G_3(t), \qquad (61)$$

and from (31) and (59) it follows that

$$G_1'(t) = -\frac{i}{(1-t^2)^{\frac{1}{2}}} G_2'(t), \quad G_3'(t) = \frac{it \tan \mu}{(1-t^2)^{\frac{1}{2}}} G_2'(t) = -t \tan \mu \, G_1'(t), \quad (62)$$

where $(1-t^2)^{\frac{1}{2}}$ is the branch, in the cut t-plane, that is positive when t is real and < 1.

Essentially the process we shall follow is to solve first for $G_2'(t)$. Certain restrictions may be immediately placed on $G_2'(t)$ in order that the solution may correspond to a physically possible flow:—

(*i*) On $|\zeta| = 1$, v is zero at $\zeta = \pm i$ and in a neighbourhood of each of these points; $g_2(\zeta)$ will be regular near $\zeta = \pm i$, and these points correspond with $t = \infty$. Since

$$\frac{dg_2}{d\zeta} = \frac{2(1-\zeta^2)}{(1+\zeta^2)^2} \frac{dG_2}{dt} = \frac{1-\zeta^2}{2\zeta^2} t^2 \frac{dG_2}{dt}, \quad . \quad . \quad . \quad (63)$$

$G_2'(t)$ must be $O(t^{-2})$ at most as $t \longrightarrow \infty$ in order that $g_2'(\zeta)$ should be finite at $\zeta = \pm i$.

(*ii*) In order that the lift force on the aerofoil should be finite, the pressure, and therefore (from (4)) w, must be integrable over the aerofoil surface, the values being equal in magnitude and opposite in sign on the two sides of the aerofoil. Over the part of the aerofoil outside the Mach cone of its vertex, w is finite. Over the part inside the Mach cone w, if not finite, must at least be integrable. It follows from this argument, and from the correspondence of the line $x/(z \tan \mu) = x_1$ with the point $t = x_1$, that $w\,(= \mathscr{R} G_3(t))$ must be integrable along the real axis in the t-plane between the extreme limits of x_1 on the aerofoil. In the sequel we shall see that $G_2'(t)$, and therefore also $G_3'(t)$, may have a factor $(t-t_1)^{n+\frac{1}{2}}$ or $(t-t_1)^m$, where t_1 is one of the limits of x_1 on the aerofoil, and n and m are integers. The preceding argument shows that n must be $\geqslant -2$, and m must be $\geqslant -1$.

(*iii*) In all cases u, v, and w must themselves be finite at $x_1 = \pm 1$. We shall see that $G_2'(t)$ may have factors $(1 \pm t)^{n+\frac{1}{2}}$, where n is an integer; $G_1'(t)$ and $G_3'(t)$ then have factors $(1 \pm t)^{n-\frac{1}{2}}$, and n must be $\geqslant 0$.

We shall consider the various cases in some detail, giving formulæ for the functions $G_1(t)$, $G_2(t)$, $G_3(t)$ in the upper half-plane ($y_1 > 0$) in each case. The velocity components u, v, w, for $r < 1$, are the real parts of these functions; in only one case do we write out explicitly expressions for these real parts in terms of x_1 and y_1. We shall not mention again the values in the lower half-plane ($y_1 < 0$), since they are covered by the fact that v is an even function, and u and w are odd functions, of y_1. The values of v on the part of $y_1 = 0$, $|x_1| \leqslant 1$, corresponding with all or part of the aerofoil surface at zero incidence, and its values on $r = 1$, are used in obtaining $G_2(t)$, and it may be verified from the final formulæ for G_2 that

they have been correctly used. It may also be verified that u and w are continuous on $r=1$, the values found in the following sections agreeing, on $r=1$, with those found in Sections 5 and 6. For this purpose it will be convenient to record the values in Sections 5 and 6 in terms of the notation of the following sections. In all the following sections we shall have

$$t_0=\sec\sigma_0=\cot\beta_0\cot\mu,\quad t_1=\sec\sigma_1=\cot\beta_1\cot\mu. \qquad (64)$$

Then, for $t_0>1$,

$$(t_0^2-1)^{-\frac{1}{2}}=\cot\sigma_0=\sin\beta_0\tan\mu_0, \qquad (65)$$

$$t_0(t_0^2-1)^{-\frac{1}{2}}\tan\mu=\tan\mu\operatorname{cosec}\sigma_0=\cos\beta_0\tan\mu_0 \qquad (66)$$

(*see* equation (48)), and similarly with suffix 1. The formulæ in (49) for the velocity components in regions 2 and 3 of Fig. 2*c* (for which $t_0>1$, $t_1>1$) become

Region	u	v	w
2	$W\alpha(t_1^2-1)^{-\frac{1}{2}}$,	$-W\alpha$,	$W\alpha t_1(t_1^2-1)^{-\frac{1}{2}}\tan\mu$
3	$-W\alpha(t_0^2-1)^{-\frac{1}{2}}$,	$-W\alpha$,	$W\alpha t_0(t_0^2-1)^{-\frac{1}{2}}\tan\mu$

(67)

The formulæ in (58) for the velocity components in region 1 of Fig. 3*c* become

$$u=W\alpha\frac{t_0-t_1}{t_1(t_0^2-1)^{\frac{1}{2}}},\quad v=-W\alpha\left[1-\frac{t_0(t_1^2-1)^{\frac{1}{2}}}{t_1(t_0^2-1)^{\frac{1}{2}}}\right],\quad w=0. \qquad (68)$$

The result most often required is the pressure on the aerofoil surface. From equation (4), $\Delta p=-\rho Ww$, and to a linear approximation the value may be taken on $y=0$ instead of on the surface of the aerofoil at incidence. For each case we shall give the value of w on the relevant part of the upper side of the plane $y=0$, designated as $y=+0$. The value on the lower side is equal in magnitude and opposite in sign.

It will appear that, in all the cases in which, at zero incidence, the aerofoil passes through the z-axis, $G_2'(t)$ has no singularity at $t=0$, so (from (62)) $G_2'(0)=0$, and it is a general result that on either side of the aerofoil w, and therefore also the pressure on the surface, reaches its extreme value at $x_1=0$ in such cases.

8. BOTH EDGES OUTSIDE THE MACH CONE OF THE VERTEX, ON OPPOSITE SIDES

For this case (Fig. 2), for $y_1>0$, u, v, and w vanish for $r\geqslant 1$ except in regions 2 and 3, where their values are given by (67). Hence, on the real axis in the t-plane,

$$\left.\begin{aligned} v = \mathscr{R}\, G_2(t) &= -W\alpha \text{ for } -t_1 < t < t_0 \\ &= 0 \text{ for } t > t_0 \text{ and } t < -t_1 \end{aligned}\right\} \quad . \quad . \quad . \quad (69)$$

The solution for $G_2(t)$ may be written down immediately. In the upper half-plane

$$G_2(t) = -\frac{W\alpha}{i\pi} \log \frac{t-t_0}{t+t_1}, \quad . \quad . \quad . \quad . \quad . \quad (70)$$

with $\arg(t-t_0) = \arg(t+t_1) = 0$ for t real and $> t_0$.

Then

$$G_2'(t) = -\frac{W\alpha(t_0+t_1)}{i\pi(t-t_0)(t+t_1)} \quad . \quad . \quad . \quad . \quad . \quad (71)$$

(so the conditions (*i*), (*ii*), (*iii*), in Section 7 are satisfied), and from (62)

$$G_1'(t) = \frac{W\alpha}{\pi}\left(\frac{1}{t-t_0} - \frac{1}{t+t_1}\right)\frac{1}{(1-t^2)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad (72)$$

$$G_3'(t) = -\frac{W\alpha \tan\mu}{\pi}\left(\frac{1}{t-t_0} - \frac{1}{t+t_1}\right)\frac{t}{(1-t^2)^{\frac{1}{2}}} \quad . \quad . \quad . \quad (73)$$

Hence (with $G_1 = G_3 = 0$ at $t = \infty$)

$$G_1(t) = -\frac{W\alpha}{\pi}\left\{\frac{1}{(t_0^2-1)^{\frac{1}{2}}}\cos^{-1}\left(\frac{1-t_0t}{t_0-t}\right) - \frac{1}{(t_1^2-1)^{\frac{1}{2}}}\cos^{-1}\left(\frac{1+t_1t}{t_1+t}\right)\right\}, \quad (74)$$

$$G_3(t) = \frac{W\alpha\tan\mu}{\pi}\left\{\frac{t_0}{(t_0^2-1)^{\frac{1}{2}}}\cos^{-1}\left(\frac{1-t_0t}{t_0-t}\right) + \frac{t_1}{(t_1^2-1)^{\frac{1}{2}}}\cos^{-1}\left(\frac{1+t_1t}{t_1+t}\right)\right\}, \quad (75)$$

on the upper half-plane.

By taking real parts we find that in the upper half-plane ($y_1 \geqslant 0$) and for $r < 1$,

$$u = -\frac{W\alpha}{\pi}\left\{\sin\beta_0\tan\mu_0\cos^{-1}\left(\frac{1-t_0x_1}{\mathscr{A}}\right) - \sin\beta_1\tan\mu_1\cos^{-1}\left(\frac{1+t_1x_1}{\mathscr{B}}\right)\right\}, \quad (76)$$

$$v = -\frac{W\alpha}{\pi}\left\{\cos^{-1}\left(\frac{(t_0t_1+1)y_1^2 - (t_0-x_1)(t_1+x_1)}{\mathscr{A}\mathscr{B}}\right)\right\}. \quad . \quad . \quad . \quad . \quad (77)$$

$$w = \frac{W\alpha}{\pi}\left\{\cos\beta_0\tan\mu_0\cos^{-1}\left(\frac{1-t_0x_1}{\mathscr{A}}\right) + \cos\beta_1\tan\mu_1\cos^{-1}\left(\frac{1+t_1x_1}{\mathscr{B}}\right)\right\}, \quad (78)$$

where
$$\left.\begin{aligned}\mathscr{A} &= [(t_0 - x_1)^2 - (t_0^2 - 1)\, y_1^2]^{\frac{1}{2}}, \\ \mathscr{B} &= [(t_1 + x_1)^2 - (t_1^2 - 1)\, y_1^2]^{\frac{1}{2}},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (79)$$

$\mathscr{A}$ and $\mathscr{B}$ are positive, and the inverse cosines are between 0 and π.

The value of w on $y = +0$, $|x_1| \geqslant 1$ is given in (49) and (67) for the regions where it is not zero. For $|x_1| \leqslant 1$, on $y = +0$,

$$w = \frac{W\alpha}{\pi}\left\{\cos\beta_0 \tan\mu_0 \cos^{-1}\left(\frac{1 - t_0 x_1}{t_0 - x_1}\right) + \cos\beta_1 \tan\mu_1 \cos^{-1}\left(\frac{1 + t_1 x_1}{t_1 + x_1}\right)\right\}. \quad (80)$$

9. BOTH EDGES OUTSIDE THE MACH CONE OF THE VERTEX, AND ON THE SAME SIDE

For this case (Fig. 3), for $y_1 > 0$, u, v, and w vanish for $r \geqslant 1$ except in regions 1 and 2. Their values in region 1 are given in (68), and in region 3 in (67). On the real axis in the t-plane

$$\left.\begin{aligned} v &= 0 \text{ for } t < -1 \text{ and for } t > t_0 \\ &= -W\alpha \text{ for } t_0 > t > t_1 \\ &= -W\alpha\left[1 - \frac{t_0 (t_1^2 - 1)^{\frac{1}{2}}}{t_1 (t_0^2 - 1)^{\frac{1}{2}}}\right] \text{ for } t_1 > t > 1.\end{aligned}\right\} \quad . \quad . \quad . \quad (81)$$

Hence $\mathscr{R}\, G_2'(t) = 0$ on the real axis for $|t| > 1$, $t \neq t_0$ or t_1.

The pressure, and therefore also w, are odd functions of y_1 and are continuous across the axis $y_1 = 0$ inside $r = 1$. Hence $w = 0$ on $y_1 = 0$, $|x_1| < 1$, *i.e.* $w = 0$ on the real axis in the t-plane for $|t| < 1$.* It follows that $\mathscr{R}\, G_3'(t) = 0$, and (from (62)) that $\mathscr{I}\, G_2'(t) = 0$, on the real axis in the t-plane for $|t| < 1$. Hence, on the real axis,

$$\mathscr{R}\, G_2'(t) = 0 \text{ for } |t| > 1,\ t \neq t_0 \text{ or } t_1,\ \mathscr{I}\, G_2'(t) = 0 \text{ for } |t| < 1, \quad (82)$$

and G_2 has finite discontinuities at t_0 and t_1. We therefore write

$$G_2'(t) = \frac{(1+t)^{a+\frac{1}{2}}(1-t)^{b+\frac{1}{2}}}{(t - t_0)(t - t_1)} G(t), \quad . \quad . \quad . \quad . \quad (83)$$

where a and b are integers and $G(t)$ is a rational function of t, real when t is real.

* The component of the gradient of v normal to the real axis is also zero.

The residues at the poles t_0 and t_1, multiplied by πi, must be equal to the discontinuities in G_2 at those points along the real axis.

If $G(t)=O(t^{-m})$ as $t\to\infty$, then from condition (*i*) in Section 7, $m\geqslant a+b+1$; from condition (*iii*), $a\geqslant 0$, $b\geqslant 0$. Hence $m\geqslant 1$; G has a singularity, which is at least a pole; if it is a pole, $a=b=0$, $m=1$, and G is of the form $constant/(t-t_2)$.

We found in Section 6 that the boundary between regions 1 and 2 contains a vortex sheet of constant strength. The vortex lines coincide with the streamlines, and so, to the present approximation, may be taken as straight lines parallel to the axis of z, and the strength of the sheet is then, with the same approximation, independent of z. It follows that the whole sheet is of constant strength, and is bounded by the streamline through the vertex of the aerofoil, *i.e.*, to the present approximation, by the axis of z. Hence u must have a finite discontinuity at $t=0$, so G must have a pole at $t=0$ and be equal to A_1/t, where A_1 is a constant. (That the singularity in G must be at $t=0$, and cannot be a higher singularity than a pole, also follows from the consideration that the pressure, and therefore w, must be continuous along the real axis in the t-plane except at t_0 and t_1.) The value of A_1 is found by equating the residue of $G_2'(t)$ at $t=t_0$ to $iW\alpha/\pi$. Hence

$$G_2'(t)=-\frac{W\alpha}{\pi}\frac{t_0(t_0-t_1)}{(t_0^2-1)^{\frac{1}{2}}}\frac{(1-t^2)^{\frac{1}{2}}}{t(t-t_0)(t-t_1)}. \quad . \quad . \quad . \quad (84)$$

From (62),

$$G_1'(t)=-\frac{W\alpha}{i\pi}\frac{t_0(t_0-t_1)}{(t_0^2-1)^{\frac{1}{2}}}\frac{1}{t(t-t_0)(t-t_1)}, \quad . \quad . \quad . \quad (85)$$

and

$$G_3'(t)=\frac{W\alpha}{i\pi}\frac{t_0(t_0-t_1)}{(t_0^2-1)^{\frac{1}{2}}}\frac{\tan\mu}{(t-t_0)(t-t_1)}. \quad . \quad . \quad . \quad (86)$$

Hence, on the upper half-plane,

$$G_1(t)=-\frac{W\alpha}{i\pi(t_0^2-1)^{\frac{1}{2}}}\left\{\log(t-t_0)-\frac{t_0}{t_1}\log(t-t_1)+\frac{t_0-t_1}{t_1}\log t\right\}, \quad . \quad . \quad . \quad (87)$$

$$G_2(t)=-\frac{W\alpha}{\pi}\left\{\cos^{-1}\left(\frac{1-t_0t}{t_0-t}\right)-\frac{t_0(t_1^2-1)^{\frac{1}{2}}}{t_1(t_0^2-1)^{\frac{1}{2}}}\cos^{-1}\left(\frac{1-t_1t}{t_1-t}\right)-\frac{t_0-t_1}{t_1(t_0^2-1)^{\frac{1}{2}}}\cosh^{-1}\left(\frac{1}{t}\right)\right\}, \quad . \quad . \quad . \quad (88)$$

$$G_3(t)=\frac{W\alpha}{i\pi}\frac{t_0\tan\mu}{(t_0^2-1)^{\frac{1}{2}}}\log\left(\frac{t-t_0}{t-t_1}\right). \quad . \quad . \quad . \quad (89)$$

The vortex sheet is of constant strength

$$\frac{2(t_0-t_1)}{t_1(t_0^2-1)^{\frac{1}{2}}}W\alpha \quad . \quad . \quad . \quad (90)$$

There is a logarithmic infinity in v at $t=0$, *i.e.* along the approximate edge of the vortex sheet, the axis of z. The linear approximation is therefore not adequate in a neighbourhood of that axis.

10. ONE EDGE OUTSIDE AND ONE EDGE INSIDE THE MACH CONE

Let the edges be swept back at angles β_0, β_1, where $\beta_0 < \frac{1}{2}\pi - \mu < \beta_1 < \frac{1}{2}\pi + \mu$. We have to distinguish two cases: (*i*) $\beta_1 > \frac{1}{2}\pi$ (Fig. 5); (*ii*) $\beta_1 < \frac{1}{2}\pi$ (Fig. 6). In the first case the aerofoil extends through the z-axis; in the second case it does not. When $\beta_1 > \frac{1}{2}\pi$ the flow in the vicinity of the edge swept back at the angle β_1 resembles that near a subsonic leading edge, and singularities may be expected there on a linear theory. If, however, $\beta_1 < \frac{1}{2}\pi$, the flow near the edge β_1 is like that near a subsonic trailing edge, and we apply the Kutta-Joukowski condition that the velocity shall be finite there; there will be a trailing vortex sheet of uniform strength downstream from this edge, with its other boundary on the z-axis. In both cases the velocities in the region 3 are those in (49) or (67).

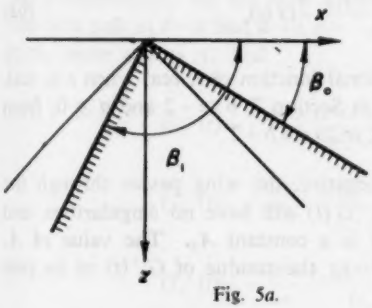

Fig. 5*a*.

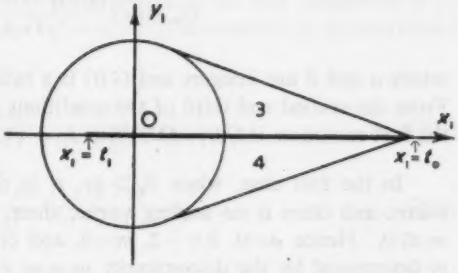

Fig. 5*b*.

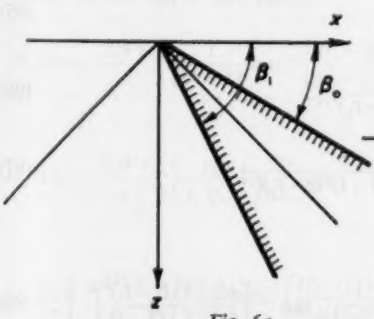

Fig. 6*a*.

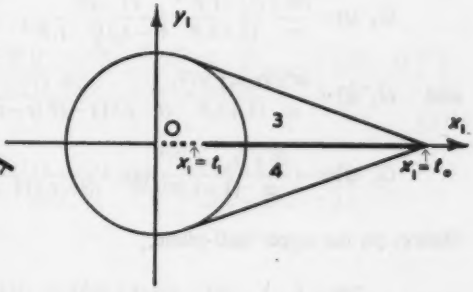

Fig. 6*b*.

The boundary conditions in the t-plane are the same in both cases; the differences in the solutions come from differences in the singularities. By the same arguments as in Section 9, on the real axis in the t-plane

$$\left.\begin{aligned} v &= -W\alpha \text{ for } t_1 < t < t_0 \\ &= 0 \text{ for } t < -1 \text{ and for } t > t_0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (91)$$

$$w = 0 \text{ for } -1 < t < t_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

Thus

$$\left.\begin{aligned} \mathscr{R}G_2'(t) &= 0 \text{ for } t < -1 \text{ and for } t > t_1,\ t \neq t_0 \\ \mathscr{I}G_2'(t) &= 0 \text{ for } -1 < t < t_1 \end{aligned}\right\} \quad . \quad . \quad (93)$$

on the real axis, and $G_2'(t)$ is of the form

$$G_2'(t) = i\frac{(1+t)^{a+\frac{1}{2}}(t-t_1)^{b+\frac{1}{2}}}{t-t_0}G(t), \quad . \quad . \quad . \quad . \quad (94)$$

where a and b are integers and $G(t)$ is a rational function of t, real when t is real. From the second and third of the conditions in Section 7, $b \geqslant -2$ and $a \geqslant 0$; from the first condition, if $G(t) = O(t^{-m})$ as $t \to \infty$, $m \geqslant a+b+2$.

In the first case, when $\beta_1 > \frac{1}{2}\pi$, t_1 is negative, the wing passes through the z-axis, and there is no trailing vortex sheet. $G(t)$ will have no singularities, and $m \leqslant 0$. Hence $a=0$, $b=-2$, $m=0$, and G is a constant A_2. The value of A_2 is determined by the discontinuity in v at $t=t_0$; the residue of $G_2'(t)$ at its pole $t=t_0$ must be $iW\alpha/\pi$. Hence

$$G_2'(t) = \frac{iW\alpha}{\pi}\frac{(t_0-t_1)^{3/2}}{(1+t_0)^{\frac{1}{2}}}\frac{(1+t)^{\frac{1}{2}}}{(t-t_0)(t-t_1)^{3/2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

and

$$G_1'(t) = \frac{W\alpha}{\pi}\frac{(t_0-t_1)^{3/2}}{(1+t_0)^{\frac{1}{2}}}\frac{1}{(t-t_0)(1-t)^{\frac{1}{2}}(t-t_1)^{3/2}}, \quad . \quad . \quad . \quad . \quad . \quad (96)$$

$$G_3'(t) = -\frac{W\alpha}{\pi}\frac{(t_0-t_1)^{3/2}}{(1+t_0)^{\frac{1}{2}}}\tan\mu\frac{t}{(t-t_0)(1-t)^{\frac{1}{2}}(t-t_1)^{3/2}}. \quad . \quad . \quad . \quad (97)$$

Hence, on the upper half-plane,

$$G_1(t) = \frac{2W\alpha}{\pi}\left\{\frac{1}{1-t_1}\left[\frac{(t_0-t_1)(1-t)}{(t_0+1)(t-t_1)}\right]^{\frac{1}{2}} - \frac{1}{(t_0^2-1)^{\frac{1}{2}}}\tan^{-1}\left[\frac{(t_0-1)(t-t_1)}{(t_0-t_1)(1-t)}\right]^{\frac{1}{2}}\right\}, \quad (98)$$

$$G_2(t) = -\frac{2W\alpha}{i\pi}\left\{\left[\frac{(t_0-t_1)(1+t)}{(t_0+1)(t-t_1)}\right]^{\frac{1}{2}} - \tanh^{-1}\left[\frac{(t_0-t_1)(1+t)}{(t_0+1)(t-t_1)}\right]^{\frac{1}{2}}\right\}, \quad . \quad . \quad . \quad (99)$$

$$G_3(t) = -\frac{2W\alpha\tan\mu}{\pi}\left\{\frac{t_1}{1-t_1}\left[\frac{(t_0-t_1)(1-t)}{(t_0+1)(t-t_1)}\right]^{\frac{1}{2}} - \frac{t_0}{(t_0^2-1)^{\frac{1}{2}}}\tan^{-1}\left[\frac{(t_0-1)(t-t_1)}{(t_0-t_1)(1-t)}\right]^{\frac{1}{2}}\right\}$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

On $y_1 = +0$, for $t_1 < x_1 < 1$ (*i.e.* for the part of the aerofoil surface, at zero incidence, inside the Mach cone)

$$w = \frac{2W\alpha}{\pi}\left\{\frac{-\cot\beta_1}{1-t_1}\left[\frac{(t_0-t_1)(1-x_1)}{(t_0+1)(x_1-t_1)}\right]^{\frac{1}{2}} + \cos\beta_0\tan\mu_0\tan^{-1}\left[\frac{(t_0-1)(x_1-t_1)}{(t_0-t_1)(1-x_1)}\right]^{\frac{1}{2}}\right\}, \quad (101)$$

$\cot\beta_1$ and t_1 being negative.

In the second case, when $\beta < \frac{1}{2}\pi$, t_1 is positive, and there is a trailing vortex sheet. $G_2'(t)$ is still given by equation (94), but the Kutta-Joukowski condition, that the velocity shall be finite at $t = t_1$, requires that $b \geqslant -1$. Hence $m \geqslant 1$, and $G(t)$ has a singularity. By the same argument as in Section 9, the only singularity in G can be a pole at $t = 0$, and $a = 0$, $b = -1$, $G(t) = A_3/t$. The constant A_3 is determined in the same way as A_2, and

$$G_1'(t) = \frac{W\alpha}{\pi}\frac{t_0(t_0-t_1)^{\frac{1}{2}}}{(t_0+1)^{\frac{1}{2}}}\frac{1}{t(t-t_0)(t-t_1)^{\frac{1}{2}}(1-t)^{\frac{1}{2}}}, \quad . \quad . \quad (102)$$

$$G_2'(t) = \frac{iW\alpha}{\pi}\frac{t_0(t_0-t_1)^{\frac{1}{2}}}{(t_0+1)^{\frac{1}{2}}}\frac{(1+t)^{\frac{1}{2}}}{t(t-t_0)(t-t_1)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad (103)$$

$$G_3'(t) = -\frac{W\alpha}{\pi}\frac{t_0(t_0-t_1)^{\frac{1}{2}}\tan\mu}{(t_0+1)^{\frac{1}{2}}}\frac{1}{(t-t_0)(t-t_1)^{\frac{1}{2}}(1-t)^{\frac{1}{2}}} \quad . \quad . \quad (104)$$

Hence, on the upper half-plane,

$$G_1(t) = -\frac{2W\alpha}{\pi}\left\{\frac{1}{(t_0^2-1)^{\frac{1}{2}}}\tan^{-1}\left[\frac{(t_0-1)(t-t_1)}{(t_0-t_1)(1-t)}\right]^{\frac{1}{2}} - \left[\frac{t_0-t_1}{t_1(t_0+1)}\right]^{\frac{1}{2}}\tan^{-1}\left[\frac{t_1(1-t)}{t-t_1}\right]^{\frac{1}{2}}\right\},$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

$$G_2(t) = \frac{2iW\alpha}{\pi}\left\{\left[\frac{t_0-t_1}{t_1(t_0+1)}\right]^{\frac{1}{2}}\tan^{-1}\left[\frac{t_1(1+t)}{t-t_1}\right]^{\frac{1}{2}} - \tanh^{-1}\left[\frac{(t_0-t_1)(1+t)}{(t_0+1)(t-t_1)}\right]^{\frac{1}{2}}\right\},$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

$$G_3(t) = \frac{2W\alpha}{\pi}\frac{t_0\tan\mu}{(t_0^2-1)^{\frac{1}{2}}}\tan^{-1}\left[\frac{(t_0-1)(t-t_1)}{(t_0-t_1)(1-t)}\right]^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

On $y_1 = +0$, for $t_1 < x_1 < 1$,

$$w = \frac{2W\alpha}{\pi} \cos\beta_0 \tan\mu_0 \tan^{-1}\left[\frac{(t_0-1)(x_1-t_1)}{(t_0-t_1)(1-x_1)}\right]^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (108)$$

The vortex sheet is of constant strength

$$2W\alpha\left[\frac{t_0-t_1}{t_1(t_0+1)}\right]^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

It is convenient at this stage to exhibit a solution for the tip of a swept-back aerofoil. Let O' (Fig. 7*a*) be the tip of an aerofoil whose vertex is O, and let OO'

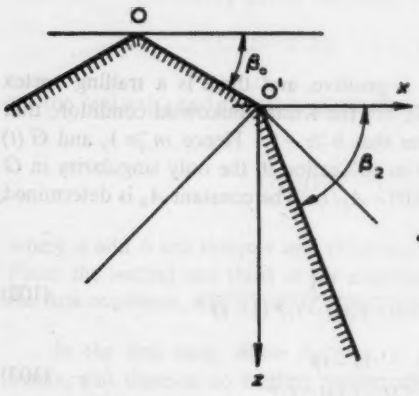

Fig. 7*a*.

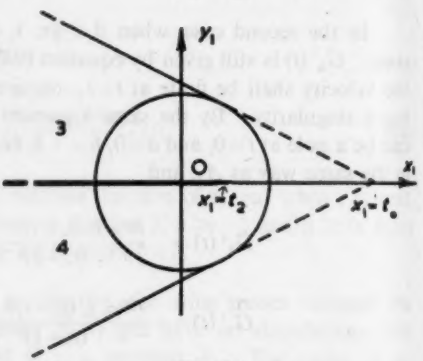

Fig. 7*b*.

be a leading edge swept back at an angle $\beta_0 < \frac{1}{2}\pi - \mu$. With O' as origin, this edge appears as a supersonic leading edge swept forward at an angle β_0. Let the other edge through O' be swept back at an angle β_2, where $\frac{1}{2}\pi - \mu < \beta_2 < \frac{1}{2}\pi + \mu$. (The cases in which $\beta_2 < \frac{1}{2}\pi - \mu$ and $\beta_2 > \frac{1}{2}\pi + \mu$ may be dealt with similarly by an extension of the analysis of Sections 8 and 9.) Thus one edge through O' is outside the Mach cone of O', and the other edge is inside it. We are for the present concerned to find the velocity field inside the portion of the Mach cone of O' that is not intersected by the Mach cone of O.

The velocity over the aerofoil between the two Mach cones is given by the formulæ in Section 5, and, in the upper half-plane, by the formula for the region 3 in (49) and (67). In the (x_1, y_1) plane, with O' as origin, these formulæ give u, v, and w in the region 3 of Fig. 7*b*, and in particular over its curved boundary. Hence, with t_0 defined as before (equation (64)), the expressions for u, v, and w for region 3

in (67) give the values on the upper side of the real axis in the t-plane for $t > t_0$ and $t < -1$ (since the relevant portion of the unit circle in the (x_1, y_1) plane transforms into these parts of the real axis in the t-plane). Over the part of the unit circle in the (x_1, y_1) plane that is not the boundary of the regions 3 and 4 the velocity components vanish, so, on the real axis in the t-plane, $u=v=w=0$ for $1 \leqslant t < t_0$. On the part of the aerofoil itself (taken at zero incidence, as usual) inside the Mach cone, $v = -W\alpha$, so if

$$t_2 = \cot\beta_2 \cot\mu, \qquad . \quad . \quad . \quad . \quad . \quad (110)$$

$v = -W\alpha$ on the real axis in the t-plane for $-1 \leqslant t < t_2$. Furthermore, for $t_1 < t < 1$, $w=0$ on the real axis, by the same reasoning as in Section 9.

There are again two cases to be considered. If $\beta_2 < \frac{1}{2}\pi$, $t_2 > 0$, the edge swept back at an angle β_2 is a subsonic leading edge, at which we expect singularities on a linear theory, and there is no vortex sheet. (This is the case illustrated in Fig. 7.) If, however, $\beta_2 > \frac{1}{2}\pi$, $t_2 < 0$, then the edge is a subsonic trailing edge, at which we apply the Kutta-Joukowski condition that the velocity shall be finite, and there is a vortex sheet extending (according to the linear approximation) along $y_1 = 0$ from $x_1 = t_2$ to $x_1 = 0$.

The boundary conditions for $G_2'(t)$ on the real axis in the t-plane are now

$$\mathscr{R}\, G_2'(t) = 0 \text{ for } t < t_2 \text{ and for } t > 1 \ (t \neq t_0), \quad \mathscr{I}\, G_2'(t) = 0 \text{ for } t_2 < t < 1. \quad (111)$$

The solution is found in the same way as before. For the first case ($\beta_2 < \frac{1}{2}\pi$, $t_2 > 0$, no vortex sheet), on the upper half-plane,

$$G_1(t) = -\frac{2iW\alpha}{\pi}\left\{\frac{1}{1+t_2}\left[\frac{(t_0-t_2)(1+t)}{(t_0-1)(t-t_2)}\right]^{\frac{1}{2}} - \frac{1}{(t_0^2-1)^{\frac{1}{2}}}\tanh^{-1}\left[\frac{(t_0+1)(t-t_2)}{(t_0-t_2)(1+t)}\right]^{\frac{1}{2}}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

$$G_2(t) = \frac{2W\alpha}{\pi}\left\{\left[\frac{(t_0-t_2)(1-t)}{(t_0-1)(t-t_2)}\right]^{\frac{1}{2}} - \tan^{-1}\left[\frac{(t_0-t_2)(1-t)}{(t_0-1)(t-t_2)}\right]^{\frac{1}{2}}\right\}, \quad . \quad . \quad . \quad (113)$$

$$G_3(t) = \frac{2iW\alpha\tan\mu}{\pi}\left\{\frac{t_2}{1+t_2}\left[\frac{(t_0-t_2)(1+t)}{(t_0-1)(t-t_2)}\right]^{\frac{1}{2}} - \frac{t_0}{(t_0^2-1)^{\frac{1}{2}}}\tanh^{-1}\left[\frac{(t_0+1)(t-t_2)}{(t_0-t_2)(1+t)}\right]^{\frac{1}{2}}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (114)$$

On $y_1 = +0$, $-1 < x_1 < t_2$ (*i.e.* on the part of the upper aerofoil surface inside the Mach cone of O'),

$$w = \frac{2W\alpha}{\pi}\left\{\frac{\cot\beta_2}{1+t_2}\left[\frac{(t_0-t_2)(1+x_1)}{(t_0-1)(t_2-x_1)}\right]^{\frac{1}{2}} + \cos\beta_0\tan\mu_0\tan^{-1}\left[\frac{(t_0+1)(t_2-x_1)}{(t_0-t_2)(1+x_1)}\right]^{\frac{1}{2}}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115)$$

For the second case ($\beta_2 > \frac{1}{2}\pi$, $t_2 < 0$, vortex sheet from $x_1 = t_2$ to $x_1 = 0$), on the upper half-plane,

$$G_1(t) = -\frac{2iW\alpha}{\pi}\left\{\left[\frac{(t_0-t_2)}{-t_2(t_0-1)}\right]^{\frac{1}{2}} \tanh^{-1}\left[\frac{-t_2(1+t)}{t-t_2}\right]^{\frac{1}{2}} - \right.$$

$$\left. -\frac{1}{(t_0^2-1)^{\frac{1}{2}}} \tanh^{-1}\left[\frac{(t_0+1)(t-t_2)}{(t_0-t_2)(1+t)}\right]^{\frac{1}{2}}\right\}, \quad . \quad (116)$$

$$G_2(t) = \frac{2W\alpha}{\pi}\left\{\left[\frac{(t_0-t_2)}{-t_2(t_0-1)}\right]^{\frac{1}{2}} \tanh^{-1}\left[\frac{-t_2(1-t)}{t-t_1}\right]^{\frac{1}{2}} - \right.$$

$$\left. -\tan^{-1}\left[\frac{(t_0-t_2)(1-t)}{(t_0-1)(t-t_2)}\right]^{\frac{1}{2}}\right\}, \quad . \quad . \quad (117)$$

$$G_3(t) = -\frac{2iW\alpha t_0 \tan\mu}{\pi(t_0^2-1)^{\frac{1}{2}}} \tanh^{-1}\left[\frac{(t_0+1)(t-t_2)}{(t_0-t_2)(1+t)}\right]^{\frac{1}{2}} . \quad . \quad . \quad . \quad . \quad . \quad (118)$$

On $y = +0$, for $-1 < x_1 < t_2$,

$$w = \frac{2W\alpha}{\pi}\cos\beta_0 \tan\mu_0 \tan^{-1}\left[\frac{(t_0+1)(t_2-x_1)}{(t_0-t_2)(1+x_1)}\right]^{\frac{1}{2}} . \quad . \quad . \quad (119)$$

The strength of the vortex sheet is

$$2W\alpha\left[\frac{t_0-t_2}{-t_2(t_0-1)}\right]^{\frac{1}{2}} . \quad . \quad . \quad . \quad . \quad (120)$$

11. BOTH EDGES INSIDE THE MACH CONE

Let the edges be swept back at angles β_0 and β_1 (Figs. 8*a*, 8*b*), where $\frac{1}{2}\pi - \mu < \beta_0 < \frac{1}{2}\pi$ and $\beta_0 < \beta_1 < \frac{1}{2}\pi + \mu$. All the velocity components, u, v, and w,

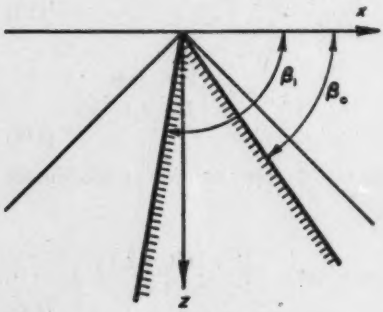

Fig. 8*a*.

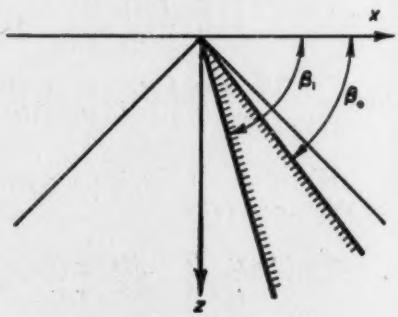

Fig. 8*b*.

are zero on the circle $r=1$ in the (x_1, y_1) plane, and therefore also on the real axis in the t-plane for $|t| \geqslant 1$. Also $v=-W\alpha$ on the real axis in the t-plane for $t_1 < t < t_0$, where t_0, t_1 are defined in (64), and, by the same reasoning as in Section 9, $w=0$ for $-1 < t < t_1$ and for $t_0 < t < 1$. Hence

$$\left.\begin{aligned} &\mathscr{R}\, G_2'(t)=0 \text{ for } t<-1, \text{ for } t_1<t<t_0, \text{ and for } t>1, \\ &\mathscr{I}\, G_2'(t)=0 \text{ for } -1<t<t_1 \text{ and for } t_0<t<1, \end{aligned}\right\} \quad . \quad (121)$$

on the real axis, and

$$G_2'(t)=(1-t)^{a+\frac{1}{2}}(1+t)^{b+\frac{1}{2}}(t-t_0)^{c+\frac{1}{2}}(t-t_1)^{d+\frac{1}{2}}G(t), \quad . \quad . \quad (122)$$

where a, b, c, d are integers and $G(t)$ is a rational function of t, real when t is real.

There are again two cases. In the first (Fig. 8*a*), $\beta_1 > \frac{1}{2}\pi$, $t_1 < 0$, the aerofoil at zero incidence extends through the z-axis, both edges are subsonic leading edges, at which singularities are to be expected on a linear theory, and there is no vortex sheet. In the second (Fig. 8*b*), $\beta_1 < \frac{1}{2}\pi$, $t_1 > 0$, the aerofoil does not extend through the z-axis, the edge swept back at an angle β_1 is a subsonic trailing edge, at which the Kutta-Joukowski condition (that the velocity should be finite) is applied, and there is a trailing vortex sheet extending (according to the linear approximation) along $y_1=0$ from $x_1=t_1$ to $x_1=0$, corresponding with the real axis in the t-plane from $t=t_1$ to $t=0$.

In the first case, $G(t)$ has no singularities; in the second it has a pole at $t=0$. The conditions on $G_2'(t)$ in Section 7 lead to the conclusions that in the first case

$$a=b=0, \quad c=d=-2, \quad G(t)=A_4, \quad . \quad . \quad . \quad . \quad (123)$$

and in the second case (since u, v, w must be finite at $t=t_1$),

$$a=b=0, \quad c=-2, \quad d=-1, \quad G(t)=C_4/t, \quad . \quad . \quad . \quad (124)$$

where A_4 and C_4 are real constants.

Hence in the first case

$$G_2'(t)=\frac{A_4(1-t^2)^{\frac{1}{2}}}{(t-t_0)^{\frac{1}{2}}(t-t_1)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad (125)$$

and, from (62),

$$G_1'(t) = -\frac{iA_4}{(t-t_0)^{\frac{1}{2}}(t-t_1)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad (126)$$

$$G_2'(t) = \frac{iA_4 t \tan\mu}{(t-t_0)^{\frac{1}{2}}(t-t_1)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad (127)$$

whereas in the second case

$$G_2'(t) = \frac{C_4(1-t^2)^{\frac{1}{2}}}{t(t-t_0)^{\frac{1}{2}}(t-t_1)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad (128)$$

$$G_1'(t) = -\frac{iC_4}{t(t-t_0)^{\frac{1}{2}}(t-t_1)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad (129)$$

$$G_3'(t) = \frac{iC_4 \tan\mu}{(t-t_0)^{\frac{1}{2}}(t-t_1)^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad (130)$$

The constant A_4 and C_4 are found, after integration, from the condition that in each case, on the real axis, $v=0$ for $|t|>1$ and $v=-W\alpha$ for $t_1<t<t_0$.

We now integrate for the first case ($\beta_1>\frac{1}{2}\pi$, $t_1<0$, no vortex sheet). $G_1(t)$ and $G_3(t)$ are expressible as elementary functions, but $G_2(t)$ is an elliptic integral. Let

$$k^2=\frac{(1-t_0)(1+t_1)}{(1+t_0)(1-t_1)}, \quad (k')^2=1-k^2=\frac{2(t_0-t_1)}{(1+t_0)(1-t_1)}, \quad . \quad . \quad (131)$$

k, k' being positive, and let K and E be the complete elliptic integrals of the first and second kind, respectively, to modulus k, and K' the complete elliptic integral of the first kind to modulus k'. We define χ by means of Jacobian elliptic functions:

$$\mathrm{sn}(\chi, k)=\left\{\frac{(1-t_1)(t-t_0)}{(1-t_0)(t-t_1)}\right\}^{\frac{1}{2}}, \quad \mathrm{cn}(\chi, k)=\left\{\frac{(t_0-t_1)(1-t)}{(1-t_0)(t-t_1)}\right\}^{\frac{1}{2}},$$

$$\mathrm{dn}(\chi, k)=\left\{\frac{(t_0-t_1)(1+t)}{(1+t_0)(t-t_1)}\right\}^{\frac{1}{2}}, \quad . \quad (132)$$

and write $E(\chi)$ for the elliptic integral of the second kind,

$$E(\chi)=\int_0^{\chi}\mathrm{dn}^2(\theta, k)\,d\theta. \quad . \quad . \quad . \quad . \quad . \quad (133)$$

Then
$$\frac{d\chi}{dt}=\frac{1}{2}\left\{\frac{(1-t_1)(1+t_0)}{(t-t_1)(t-t_0)(1-t^2)}\right\}^{\frac{1}{2}}, \quad . \quad . \quad . \quad (134)$$

and
$$\frac{dE(\chi)}{dt}=\frac{1}{2}\left\{\frac{(1-t_1)(t_0-t_1)^2(1+t)}{(1+t_0)(1-t)(t-t_1)^3(t-t_0)}\right\}^{\frac{1}{2}}. \quad . \quad . \quad (135)$$

The values $t=1, t_0, t_1, -1$ correspond with $\chi=K, 0, iK', K+iK'$, respectively.

On the upper half-plane the integral for G_2 that vanishes at $t=1$ is

$$G_2(t)=\frac{2A_4(1+t_0)^{\frac{1}{2}}(1-t_1)^{\frac{1}{2}}}{(t_0-t_1)^2}\left\{2E-(k')^2K-2E(\chi)+(k')^2\chi\right\}-\frac{2A_4}{t_0-t_1}\left\{\frac{1-t^2}{(t-t_0)(t-t_1)}\right\}^{\frac{1}{2}}.$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (136)$$

When $t_0>t>t_1$, the last term on the right, χ, and $E(\chi)$ are all purely imaginary, so

$$v=-W\alpha=\frac{2A_4(1+t_0)^{\frac{1}{2}}(1-t_1)^{\frac{1}{2}}}{(t_0-t_1)^2}\left[2E-(k')^2K\right], \quad . \quad . \quad (137)$$

and
$$A_4=-\frac{W\alpha(t_0-t_1)^2}{2(1+t_0)^{\frac{1}{2}}(1-t_1)^{\frac{1}{2}}[2E-(k')^2K]}. \quad . \quad . \quad . \quad . \quad (138)$$

It may now be checked that $v=0$ when t is real and $|t|>1$.

The expression $2E-(k')^2K$ may be transformed by a transformation due to Gauss. A new modulus k_1 is introduced, where

$$1-k_1^2=(k_1')^2=\left(\frac{1-k}{1+k}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad (139)$$

and the variable of integration ψ in the defining integrals for E and K,

$$E=\int_0^{\pi/2}(1-k^2\sin^2\psi)^{\frac{1}{2}}\,d\psi, \quad K=\int_0^{\pi/2}(1-k^2\sin^2\psi)^{-\frac{1}{2}}\,d\psi, \quad . \quad . \quad (140)$$

is changed by writing

$$\sin\psi_1=\frac{(1+k)\sin\psi}{1+k\sin^2\psi}, \quad \sin\psi=\frac{1-(1-k_1^2\sin^2\psi_1)^{\frac{1}{2}}}{(1-k_1')\sin\psi_1}, \quad . \quad . \quad (141)$$

and it may be proved that

$$2E-(k')^2K=(1+k)E_1, \quad . \quad . \quad . \quad . \quad . \quad (142)$$

where E_1 is the complete elliptic integral of the second kind to modulus k_1.

Integrals for G_1 and G_3 on the upper half-plane whose real parts vanish when t is real and $|t|>1$ are

$$G_1(t)=\frac{2iA_4}{(t_0-t_1)^2}\left\{\left(\frac{t-t_0}{t-t_1}\right)^{\frac{1}{2}}+\left(\frac{t-t_1}{t-t_0}\right)^{\frac{1}{2}}\right\}, \quad . \quad . \quad . \quad (143)$$

$$G_3(t)=-\frac{2iA_4\tan\mu}{(t_0-t_1)^2}\left\{t_1\left(\frac{t-t_0}{t-t_1}\right)^{\frac{1}{2}}+t_0\left(\frac{t-t_1}{t-t_0}\right)^{\frac{1}{2}}\right\}. \quad . \quad . \quad (144)$$

Hence, on $y_1=+0$, for $t_1<x_1<t_0$ (t_1 being negative),

$$w=\frac{W\alpha\tan\mu}{(1+t_0)^{\frac{1}{2}}(1-t_1)^{\frac{1}{2}}[2E-(k')^2K]}\left\{t_0\left(\frac{x_1-t_1}{t_0-x_1}\right)^{\frac{1}{2}}-t_1\left(\frac{t_0-x_1}{x_1-t_1}\right)^{\frac{1}{2}}\right\} \quad . \quad (145)$$

In the special case when the aerofoil is symmetrical about the z-axis, $t_1=-t_0$, and

$$k=\frac{1-t_0}{1+t_0}, \quad k_1^2=1-t_0^2, \quad A_4=-\frac{W\alpha t_0^2}{E_1}. \quad . \quad . \quad . \quad (146)$$

In this case the expression (145) becomes

$$w=\frac{W\alpha t_0^2\tan\mu}{E_1(t_0^2-x_1^2)^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad (147)$$

We turn now to the second case ($\beta_1<\frac{1}{2}\pi$, $t_1>0$, a vortex sheet on $y_1=0$ between $x_1=t_1$ and $x_1=0$), and integrate equations (128), (129), and (130). Again $G_1(t)$ and $G_3(t)$ are expressible as elementary functions, whereas $G_2(t)$ is an elliptic integral. To integrate $G_2'(t)$, k, K, E, χ, and $E(\chi)$ are introduced as before (equations (131), (132), (133), (140)). In addition, let

$$m=\left(\frac{1}{\mathrm{sn}^2\chi}\right)_{t=0}=\frac{t_1(1-t_0)}{t_0(1-t_1)}, \quad . \quad . \quad . \quad . \quad . \quad (148)$$

so that

$$1-m=\frac{t_0-t_1}{t_0(1-t_1)}, \quad \frac{k^2}{m}-1=\frac{t_0-t_1}{t_1(1+t_0)}, \quad . \quad . \quad (149)$$

and $1>k^2>m>0$. Also introduce the elliptic integral of the third kind

$$F(\chi)=(1-m)\int_0^{\chi}\frac{d\psi}{1-m\,\mathrm{sn}^2\psi}, \quad . \quad . \quad . \quad . \quad . \quad (150)$$

for which

$$\frac{dF(\chi)}{dt} = \frac{1}{2t}\left\{\frac{(1+t_0)(t-t_1)}{(1-t_1)(t-t_0)(1-t^2)}\right\}^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (151)$$

This elliptic integral has not been tabulated (Fletcher, Miller, and Rosenhead, *An Index of Mathematical Tables, London*, 1946, p. 318), but is expressible in terms of theta functions. The corresponding complete elliptic integral will be denoted by F, so that

$$F=(1-m)\int_0^{K} \frac{d\psi}{1-m\,\mathrm{sn}^2\,\psi} = (1-m)\int_0^{\pi/2} \frac{d\phi}{(1-m\sin^2\phi)(1-k^2\sin^2\phi)^{\frac{1}{2}}} \quad . \quad (152)$$

Then on the upper half-plane the integral for G_2 that vanishes at $t=1$ is

$$G_2(t)=\frac{2C_4(1+t_0)^{\frac{1}{2}}(1-t_1)^{\frac{1}{2}}}{t_0(t_0-t_1)}\left\{E-E(\chi)+\left(\frac{k^2}{m}-1\right)\left[K-F-\chi+F(\chi)\right]\right\} -$$

$$-\frac{2C_4}{t_0}\left\{\frac{1-t^2}{(t-t_1)(t-t_0)}\right\}^{\frac{1}{2}} . \quad (153)$$

When $t_0 > t > t_1$, the last term on the right, χ, $E(\chi)$, and $F(\chi)$ are all purely imaginary, so

$$v=-W\alpha=\frac{2C_4(1+t_0)^{\frac{1}{2}}(1-t_1)^{\frac{1}{2}}}{t_0(t_0-t_1)}\left\{E+\left(\frac{k^2}{m}-1\right)\left(K-F\right)\right\} \quad . \quad . \quad (154)$$

and

$$-\frac{2C_4}{t_0-t_1}=\frac{W\alpha t_0}{(1+t_0)^{\frac{1}{2}}(1-t_1)^{\frac{1}{2}}\{E+(k^2/m-1)(K-F)\}} \quad . \quad . \quad . \quad (155)$$

It may now be checked that $v=0$ when t is real and $|t|>1$.

Integrals for G_1 and G_3 in the upper half-plane whose real parts vanish when t is real and $|t|>1$ are

$$G_1(t)=\frac{2iC_4}{t_0(t_0-t_1)}\left\{\left(\frac{t-t_1}{t-t_0}\right)^{\frac{1}{2}}+\frac{t_0-t_1}{(t_0t_1)^{\frac{1}{2}}}\tanh^{-1}\left[\frac{t_1(t-t_0)}{t_0(t-t_1)}\right]^{\frac{1}{2}}\right\} \quad . \quad (156)$$

$$G_3(t)=-\frac{2iC_4\tan\mu}{t_0-t_1}\left(\frac{t-t_1}{t-t_0}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (157)$$

Hence, on $y_1=+0$, for $t_1<x_1<t_0$,

$$w=-\frac{2C_4\tan\mu}{t_0-t_1}\left(\frac{x_1-t_1}{t_0-x_1}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (158)$$

The strength of the vortex sheet is

$$\frac{2\pi \mid C_4 \mid}{t_0^{\frac{3}{2}} t_1^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (159)$$

As $\beta_0 - \beta_1 \to 0$, $t_0 - t_1 \to 0$, $m \to 1$, $k \to 1$, $E \to 1$, $(k^2/m - 1)(K - F) \to 0$,

and

$$-\frac{2C_4 \tan \mu}{t_0 - t_1} \to \frac{W\alpha t_0 \tan \mu}{(1 - t_0^2)^{\frac{1}{2}}} = \frac{W\alpha \cos \beta_0}{(1 - M^2 \cos^2 \beta_0)^{\frac{1}{2}}}. \quad . \quad . \quad (160)$$

Also

$$\frac{x_1 - t_1}{t_0 - x_1} = \frac{l_1 \sin \beta_0}{l_0 \sin \beta_1}, \quad . \quad . \quad . \quad . \quad . \quad (161)$$

where

$$l_0 = z \cos \beta_0 - x \sin \beta_0, \quad l_1 = x \sin \beta_1 - z \cos \beta_1. \quad . \quad . \quad . \quad (162)$$

As $\beta_0 - \beta_1 \to 0$, $\sin \beta_0 / \sin \beta_1 \to 1$, and the aerofoil, at large distances from the vertex, of the order of $(\beta_0 - \beta_1)^{-1}$, becomes an aerofoil of infinite span yawed at an angle $\beta_0 > \frac{1}{2}\pi - \mu$; on $y = 0$, l_1/l_0 is the ratio of the distance of a point from the trailing edge to its distance from the leading edge; and the formula (158) for w on $y_1 = +0$ for $t_1 < x_1 < t_0$ becomes the usual formula for an infinite yawed plane wing of constant chord. The strength of the vortex sheet tends to zero with $t_0 - t_1$.

12. FLOW OVER MORE GENERAL PLANE AEROFOILS BY FINITE COMBINATIONS OF CONICAL FIELDS

The solutions in Sections 8, 9, and 10 may be combined to give the flow over certain extended classes of plane aerofoils at small incidences. We shall consider here only aerofoils the flow over which can be obtained by the combination of a finite number of conical fields. All supersonic leading edges, and all subsonic edges, of such aerofoils must be straight. The occurrence of a straight subsonic edge sets a limit to the extent of such an aerofoil; such an edge is inside the Mach cone of its upstream end, and a conical field (or finite combination of conical fields) cannot be found with the downstream end as origin to give correctly the flow over an adjacent edge on the downstream side. More extended classes of aerofoils, including aerofoils with curved leading edges, which may be obtained by superposing continuous distributions of conical fields—*i.e.*, by integrating conical-field solutions—are not considered here.

The method of combination may be explained by considering the plane aerofoil shown in Fig. 9. Fig. 9*a* shows the aerofoil at zero incidence, in the plane $y = 0$; to a linear approximation it is also the projection on to the plane $y = 0$ of the surface of the aerofoil at a small incidence α. *OP*, *OQ* are straight supersonic leading edges;

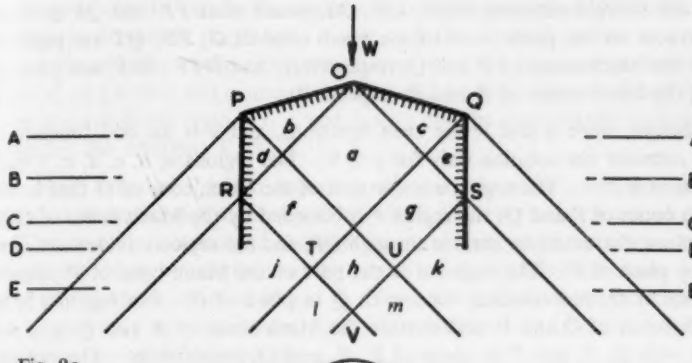

Fig. 9*a*.

Fig. 9*b*. SECTION ON AA'

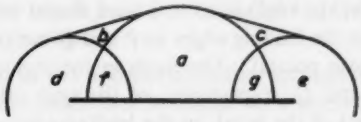

Fig. 9*c*. SECTION ON BB'

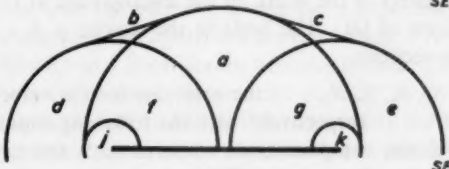

Fig. 9*d*. SECTION ON CC'

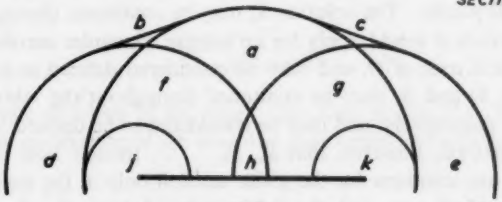

Fig. 9*e*. SECTION ON DD'

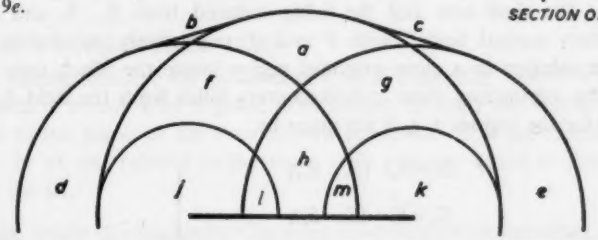

Fig. 9*f*. SECTION ON EE'

PR, *QS* are straight subsonic edges; *OR*, *OS*, which meet *PR* and *QS* in *R* and *S*, are the traces on the plane $y=0$ of the Mach cone of *O*; *PU*, *QT* are parts of the traces of the Mach cones of *P* and *Q*, respectively; and *RTV*, *SUV* are parts of the traces of the Mach cones of *R* and *S*, respectively.

As before, since u and w are even functions, and v is an odd function, of y, we need consider the solution only for $y \geqslant 0$. The regions *a*, *b*, *c*, *d*, *e*, *f*, etc., are all regions in $y \geqslant 0$. The region *a* is the part of the Mach cone of *O* that is outside the Mach cones of *P* and *Q*; the region *b* is bounded by the Mach cones of *O* and *P* and the plane characteristic surface through *OP*, and the region *c* is defined similarly with *Q* in place of *P*. The region *d* is the part of the Mach cone of *P* outside the Mach cone of *O*, and similarly for *e* with *Q* in place of *P*. The region *f* is inside the Mach cones of *O* and *P*, and outside the Mach cones of *R* and *Q*; *g* is defined similarly with *Q*, *S*, and *P* in place of *P*, *R*, and *Q*, respectively. The region *h* is inside the Mach cones of *P* and *Q*, and outside the Mach cones of *R* and *S*. Sections through the regions are shown in Figs. 9*b*-9*f*.

The velocity fields in the regions *a*, *b*, *c* were found in Sections 5 and 8 (for the effects of the breaks in the leading edges at *P* and *Q* are confined to the interiors of the Mach cones of these points). The fields in the regions *d* and *e* are conical fields with origins at *P* and *Q*, respectively, of the type calculated at the end of Section 10 (since the effect of the break in the leading edge at *O* is confined to the inside of the Mach cone of *O*). The fields in the regions *a*, *b*, *c*, *d*, *e* are therefore found as in previous sections.

Now denote by S_a, S_b, S_c, S_d, . . . the solutions for the velocity components in the regions *a*, *b*, *c*, *d*, . . . , respectively, with the following stipulations. In S_b and S_c, the values of the velocity components are constant, so S_b and S_c may be considered to be defined at all points. The solution S_a may be continued throughout the whole of the region in which it would apply for an infinite triangular aerofoil, which is the interior of the Mach cone of *O*, and may be considered defined at all points of that region. Similarly S_d and S_e may be continued throughout the whole of the Mach cones of *P* and *Q*, respectively, and may be considered to be defined in those regions. (It is to be understood, however, that S_a, S_b, . . . , though now defined in more extended regions, are solutions for the given aerofoil only in the regions *a*, *b*, . . . , respectively). We shall now call the fields deduced from $S_b - S_d$ and $S_c - S_e$ the "complementary conical fields" with *P* and *Q*, respectively, as origins, and shall show that the solution in a more extended region inside the Mach cone of *O* may be obtained by subtracting these complementary fields from the field S_a. In fact, the solutions for the regions *f*, *g*, *h* are given by

$$\left.\begin{aligned} S_f &= S_a - (S_b - S_d), \\ S_g &= S_a - (S_c - S_e), \\ S_h &= S_a - (S_b - S_d) - (S_c - S_e). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (163)$$

These solutions clearly satisfy the approximate (linearised) field equations. They also satisfy the boundary condition $v = -W\alpha$ on the aerofoil surface at $y=0$, and they make the velocity components continuous over the common boundaries of f with a and d, of g with a and e, and of h with f and g. The value of w, and therefore of Δp, on $y = +0$ in the regions f, g, and h may now be immediately deduced from the values for the solutions S_a, S_b, S_c, S_d, S_e.

The Mach cones of R and S include portions of the plane $y=0$ that are not on the projection of the aerofoil surface, and on which, to preserve continuity of pressure, $w=0$. This condition cannot be satisfied by combinations of the above type that also satisfy the other boundary conditions, and so no such combinations are valid anywhere in the Mach cones of R and S. Extended regions of the aerofoil surface may be treated only by superposing continuous distributions of conical fields—*i.e.*, by integrating conical-field solutions.

The foregoing solution yields the pressure over the whole aerofoil surface if the trailing edge of the aerofoil is any line (not necessarily straight, or in straight segments) that lies in the region bounded by $OPRVSQO$ in Fig. 9*a* and whose angle of sweep nowhere exceeds $\frac{1}{2}\pi - \mu$.

The solutions found in this and the preceding sections may be used to calculate the resultant aerodynamic forces and moments on a given aerofoil. The lift force, parallel to the y-axis, and the moment about the z-axis, may be calculated by integrating the pressures over the upper and lower surfaces. At a subsonic leading edge, however, the velocity on a linear theory becomes infinite; if r_1 is normal distance from the edge in the aerofoil plane, the velocity $\sim Dr_1^{-\frac{1}{2}}$ as $r_1 \to 0$ (where D is a constant); there is a suction force on the leading edge, acting normally outwards from the edge and in the aerofoil plane, whose magnitude, per unit length of the edge, is

$$\frac{\pi\rho D^2}{(1-M^2\cos^2\beta)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad (164)$$

where β is the angle of sweep of the edge. (*See* Refs. 8 and 27.) This suction must be included in the calculation of the drag and side forces and yawing moment on the aerofoil.

13. PLANE VANES IN FREE JETS

In this section we solve the problem of a plane triangular vane in a semi-infinite free jet, which is bounded by a plane vortex sheet through the vertex of the vane and normal to the plane of the vane at zero incidence. The gas outside the jet is assumed to be at rest relative to the vane, with pressure equal to the free-stream pressure in the jet.

With the origin of rectangular Cartesian co-ordinates at the vertex of the vane, the jet is taken to be in the half-space $x > 0$, with its boundary on $x=0$. The vane

is at a small incidence α; when $\alpha=0$, the vane is in the plane $y=0$. The undisturbed gas velocity in the jet is W along the axis of z.

On a linear theory the displacement of the jet boundary, as well as the incidence of the vane, may be neglected in formulating the boundary conditions. The pressure must be continuous on the jet boundary, so, on a linear theory, w must be zero there. The boundary conditions are that on $y=0$, $x>0$,

$$\left.\begin{aligned} &v=-W\alpha \text{ on the projection of the vane,}\\ \text{and}\quad &w=0 \text{ elsewhere,}\end{aligned}\right\} \quad . \quad . \quad . \quad (165)$$

and also $$w=0 \text{ on } x=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (166)$$

We shall consider only vanes having one swept-back leading edge outside the jet, and the other inside the jet, so that the vane passes through the vortex sheet, $x=0$. We shall solve for the disturbance velocity (u, v, w) in the half-space $x>0$. In the (x_1, y_1) plane, since v is an even function, and u and w are odd functions, of y_1, it will suffice to find the solution in the upper right-hand quadrant.

Let the leading edge inside the jet be swept back at an angle β_o, and put $t_o=\cot\beta_o\cot\mu$ as usual. We have two cases to consider: one when $\beta_o<\frac{1}{2}\pi-\mu$, $t_o>1$, and the leading edge is outside the Mach cone of the origin, and the other when $\frac{1}{2}\pi>\beta_o>\frac{1}{2}\pi-\mu$, $0<t_o<1$, and the edge is inside the Mach cone.

The (x_1, y_1) plane is shown (for $x_1\geqslant 0$) for the first case in Fig. 10. For $r\geqslant 1$, the velocity components are zero outside region 3, and in region 3 they have the same values as in (67). We note that the condition $w=0$ on $x_1=0$ is satisfied for $r\geqslant 1$.

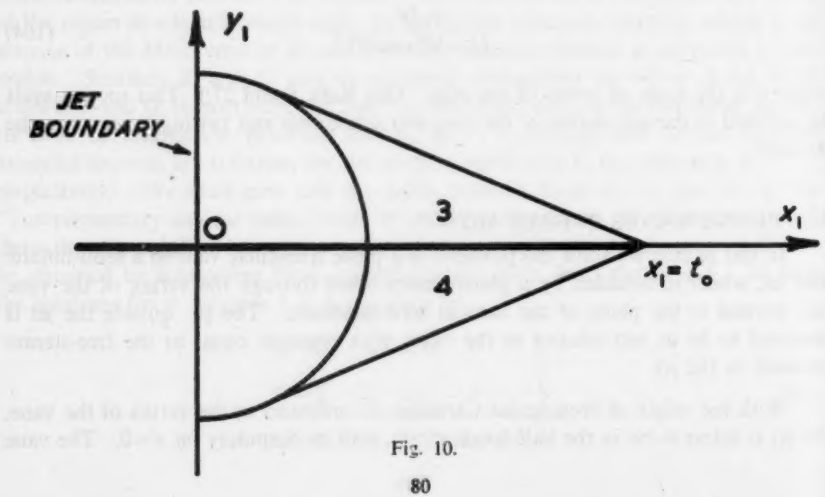

Fig. 10.

To find u, v, and w for $r \leqslant 1$, we transform to the t-plane, and find the solution in the upper right-hand quadrant of that plane. Consider the boundary conditions in the t-plane. From the values on $r=1$, we have that u, v, and w vanish on the real axis for $t > t_0$ and have the values in (67) (region 3) for $t_0 > t > 1$. From (165), $v = -W\alpha$ for $0 < t \leqslant 1$. Hence, in particular,

$$\left.\begin{aligned} v &= -W\alpha \quad \text{for } 0 < t < t_0 \\ &= 0 \qquad \text{for } t > t_0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (167)$$

on the real axis. The condition $w=0$ on $x_1=0$ becomes, for $r \leqslant 1$, $\mathscr{R} G_3(t)=0$ on the imaginary axis. Hence, on the imaginary axis, $\mathscr{I} G_3'(t)=0$, and, from (62), $\mathscr{I} G_2'(t)=0$, so v is constant; since $v=0$ when $t=i\infty$, $v=0$ everywhere on the imaginary axis.

The boundary conditions for $G_2'(t)$ are

$$\left.\begin{aligned} &\mathscr{R} G_2'(t)=0 \text{ on the real axis for } t>0\,(t \neq t_0) \\ &\mathscr{I} G_2'(t)=0 \text{ on the imaginary axis.} \end{aligned}\right\} \quad . \quad . \quad (168)$$

The first condition in Section 7 still applies, and $G_2'(t)$ is $O(t^{-2})$ at most as $t \to \infty$. The discontinuity at $t=t_0$ in the value of v along the real axis shows that $G_2'(t)$ has a pole there with residue $iW\alpha/\pi$. There is also a pole at the origin, since $v=-W\alpha$ when the origin is approached along the real axis and v is zero on the imaginary axis; the residue at the origin is $-2iW\alpha/\pi$. $G_2'(t)$ has no other singularities in the upper right-hand quadrant of the t-plane or on its boundary.

The solution for $G_2'(t)$ and $G_2(t)$ with these conditions is unique. $G_2'(t)$ is an odd function of t, the solution being

$$G_2'(t) = \frac{2iW\alpha t_0^2}{\pi t\,(t^2-t_0^2)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (169)$$

$$G_2(t) = -\frac{W\alpha}{\pi i}\log\left(\frac{t^2-t_0^2}{t^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (170)$$

From (62)

$$G_1'(t) = \frac{2W\alpha t_0^2}{\pi t\,(t^2-t_0^2)\,(1-t^2)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (171)$$

$$G_3'(t) = -\frac{2W\alpha t_0^2 \tan\mu}{\pi\,(t^2-t_0^2)\,(1-t^2)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (172)$$

whence, in the upper right-hand quadrant,

$$G_1(t) = -\frac{2W\alpha}{\pi}\left\{\frac{1}{(t_0^2-1)^{\frac{1}{2}}}\sin^{-1}\left(\frac{t_0^2-1}{t_0^2-t^2}\right)^{\frac{1}{2}} - \cosh^{-1}\left(\frac{1}{t}\right)\right\}. \quad . \quad (173)$$

$$G_3(t) = \frac{2W\alpha t_0 \tan\mu}{\pi(t_0^2-1)^{\frac{1}{2}}}\sin^{-1}\left[t\left(\frac{t_0^2-1}{t_0^2-t^2}\right)^{\frac{1}{2}}\right]. \quad . \quad . \quad . \quad . \quad (174)$$

It may be verified that the boundary conditions on u, v, and w are all satisfied.

On $y=+0$, for $0<x_1<1$,

$$w = \frac{2W\alpha}{\pi}\cos\beta_0 \tan\mu_0 \sin^{-1}\left[x_1\left(\frac{t_0^2-1}{t_0^2-x_1^2}\right)^{\frac{1}{2}}\right]. \quad . \quad . \quad (175)$$

This solution may be used in combination with other conical-field solutions to give the flow over certain vanes of finite span, as explained in Section 12.

Both in this solution, and in the solution following for the second case, there is a logarithmic infinity in u at $t=0$, corresponding with the axis of z, where the vane crosses the boundary of the jet. The linear approximation is therefore not adequate in a neighbourhood of that line.

In the second case, when $\frac{1}{2}\pi > \beta_0 > \frac{1}{2}\pi - \mu$, $0<t_0<1$, $v=w=0$ along the imaginary axis in the t-plane as before. Along the real axis, the known values are now

$$\left.\begin{aligned} v &= -W\alpha \quad \text{for } 0<t<t_0 \\ &= 0 \quad\quad \text{for } t>1, \\ u=w &= 0 \quad\quad \text{for } t>t_0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (176)$$

Thus, on the real axis, $\mathscr{R}\,G_2'(t)=0$ for $0<t<t_0$ and for $t>1$,

$$\mathscr{I}\,G_2'(t)=0 \text{ for } t_0<t<1, \quad . \quad . \quad (177)$$

and on the imaginary axis, $\mathscr{I}\,G_2'(t)=0$.

In the upper right-hand quadrant the solution for $G_2'(t)$ (satisfying the relevant conditions in Section 7) and for $G_2(t)$ is

$$G_2'(t) = \frac{2W\alpha t_0^3}{i\pi}\frac{(1-t^2)^{\frac{1}{2}}}{t(t_0^2-t^2)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (178)$$

$$G_2(t) = \frac{2W\alpha}{i\pi}\left\{t_0\left(\frac{1-t^2}{t_0^2-t^2}\right)^{\frac{1}{2}} - \tanh^{-1}\left[\frac{t_0(1-t^2)^{\frac{1}{2}}}{(t_0^2-t^2)^{\frac{1}{2}}}\right]\right\}. \quad . \quad (179)$$

Hence

$$G_1'(t) = -\frac{2W\alpha t_0^3}{\pi t (t_0^2 - t^2)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (180)$$

$$G_3'(t) = \frac{2W\alpha t_0^3 \tan\mu}{\pi (t_0^2 - t^2)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$$

$$G_1(t) = -\frac{2W\alpha}{\pi}\left\{\left(1-\frac{t^2}{t_0^2}\right)^{-\frac{1}{2}} - \tanh^{-1}\left(1-\frac{t^2}{t_0^2}\right)^{\frac{1}{2}}\right\}, \quad . \quad . \quad . \quad (182)$$

$$G_3(t) = \frac{2W\alpha t_0 \tan\mu}{\pi}\frac{t}{(t_0^2 - t^2)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (183)$$

and on $y = +0$, for $0 < x_1 < t_0$,

$$w = \frac{2W\alpha \cot\beta_0}{\pi}\frac{x_1}{(t_0^2 - x_1^2)^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (184)$$

REFERENCES

A. Conical Fields

1. BROWNE, S. H., FRIEDMAN, L., and HODES, I. (1948). A Wing-Body Problem in a Supersonic Conical Flow. *Journal of the Aeronautical Sciences*, 15, 1948, 443-452.
2. BESCHKINE, L. (1947). Ecoulements Coniques Supersoniques. Les Cahiers d'Aérodynamique, Groupes d'Etudes Aéronautiques de la Faculté des Sciences, Université de Paris, No. 6, 1947, 24-38; Forces Aérodynamiques Agissant sur les Surfaces Portantes aux Vitesses Supersoniques, *ibid.*, 39-50.
3. BUSEMANN, A. (1935). Aerodynamischer Auftrieb bei Uberschallgeschwindigkeit. Atti dei Convegni 5, Reale Accademia d'Italia (V Volta Congress), Rome, 1936, pp. 328-360; Luftfahrtforschung, 12, 1935, 210-219.
4. BUSEMANN, A. (1943). Infinitesimale kegelige Uberschallströmung. Schriften der Deutschen Akademie der Luftfahrtforschung, 7B, 1943, 105-122.
5. GERMAIN, P. (1948). Applications of the Theory of Conical Motion to Supersonic Dynamics. VII International Congress of Applied Mechanics, 1948. (Unpublished).
6. HAYES, W. D. (1946). Linearized Conical Supersonic Flow. VI International Congress of Applied Mechanics, 1946. (Unpublished).
7. HAYES, W. D., BROWNE, S. H., and LEW, R. J. (1947). Linearized Theory of Conical Supersonic Flow, with Application to Triangular Wings. North American Aviation, Inc., Report No. NA-46-818, 1947. (Unpublished).
8. HAYES, W. D. (1947). Linearized Supersonic Flow. Ph.D. Thesis, California Institute of Technology, 1947; North American Aviation, Inc., Report No. AL-222. (Unpublished).
9. LAGERSTROM, P. A. (1948). Linearized Supersonic Theory of Conical Wings. N.A.C.A. Technical Note 1685, 1948.

10. LAGERSTROM, P. A., and GRAHAM, M. E. Downwash and Sidewash induced by Three-dimensional Lifting Wings in Supersonic Flow. Douglas Aircraft Company Inc., Report No. SM-13007. (Unpublished).

11. LAGERSTROM, P. A., and GRAHAM, M. E. (1948). Linearized Theory of Supersonic Control Surfaces. Douglas Aircraft Company Inc. Report No. SM-13060. (Unpublished). For abbreviated version see *Journal of the Aeronautical Sciences*, 16 (1949), 31-34.

12. LAGERSTROM, P. A., and GRAHAM, M. E. Low Aspect Ratio Rectangular Wings in Supersonic Flow. Douglas Aircraft Company Inc., Report No. SM-13110. (Unpublished).

13. LAPORTE, O., and BARTELS, R. C. F. (1948). An Investigation of the Exact Solutions of the Linearized Equations for the Flow past Conical Bodies. Bumblebee Series Report No. 75, Engineering Research Institute, University of Michigan, 1948. (Unpublished; abstract in *Aero. Engineering Review*, February 1949, p. 48).

14. LAPORTE, O., and BARTELS, R. C. F. (1948). Supersonic Flow past a Delta Wing at Angles of Attack and Yaw. Engineering Research Institute, University of Michigan, 1948. (Unpublished).

15. PORITSKY, H. (1949). Linearized Compressible Flow. *Quarterly of Applied Mathematics*, 6, 1949, 389-403.

16. ROBERTS, R. C. (1948). Note on the Lift of a Triangular Wing at Supersonic Speeds. *Journal of Mathematics and Physics*, 27, 1948, 49-55.

17. ROPER, G. M. (1948). The Flat Delta Wing at Incidence, at Supersonic Speeds, when the Leading Edges lie outside the Mach Cone of the Vertex. *Quarterly Journal of Mechanics and Applied Mathematics*, 1, 1948, 327-343.

18. SNOW, R. M. (1948). Aerodynamics of Thin Quadrilateral Wings at Supersonic Speeds. *Quarterly of Applied Mathematics*, 5, 1948, 417-428.

19. STEWART, H. J. (1946). The Lift of a Delta Wing at Supersonic Speeds. *Quarterly of Applied Mathematics*, 4, 1946, 246-254.

20. WARD, G. N. (1946). The Pressure Distribution on some Flat Laminar Aerofoils at Incidence at Supersonic Speeds. R. & M. 2206, 1946.

B. Other Methods

21. GUNN, J. C. (1947). Linearized Supersonic Aerofoil Theory. *Philosophical Transactions of the Royal Society*, 240, 1947, 327-373.

22. HAYES, W. D. (1947). Linearized Supersonic Flow. Ph.D. Thesis, California Institute of Technology, 1947; North American Aviation, Inc., Report No. AL-222. (Unpublished).

23. LIGHTHILL, M. J. (1944). The Supersonic Theory of Wings of Finite Span. R. & M. 2001, 1944.

24. ROBINSON, A. (1946). Aerofoil Theory of a Flat Delta Wing at Supersonic Speeds. A.R.C. 10222, 1946. (Unpublished).

25. SCHLICHTING, H. (1936). Tragflügeltheorie bei Uberschallgeschwindigkeit. Luftfahrtforschung, 13 (1936), 320-335.

26. TAUNT, D. R., and WARD, G. N. (1946). Wings of Finite Aspect Ratio at Supersonic Velocities. R. & M. 2421 (1946).

27. WARD, G. N. (1949). Supersonic Flow past Thin Wings, 1: General Theory. *Quarterly Journal of Mechanics and Applied Mathematics*, Part II, Vol. II, June 1949, p. 136.